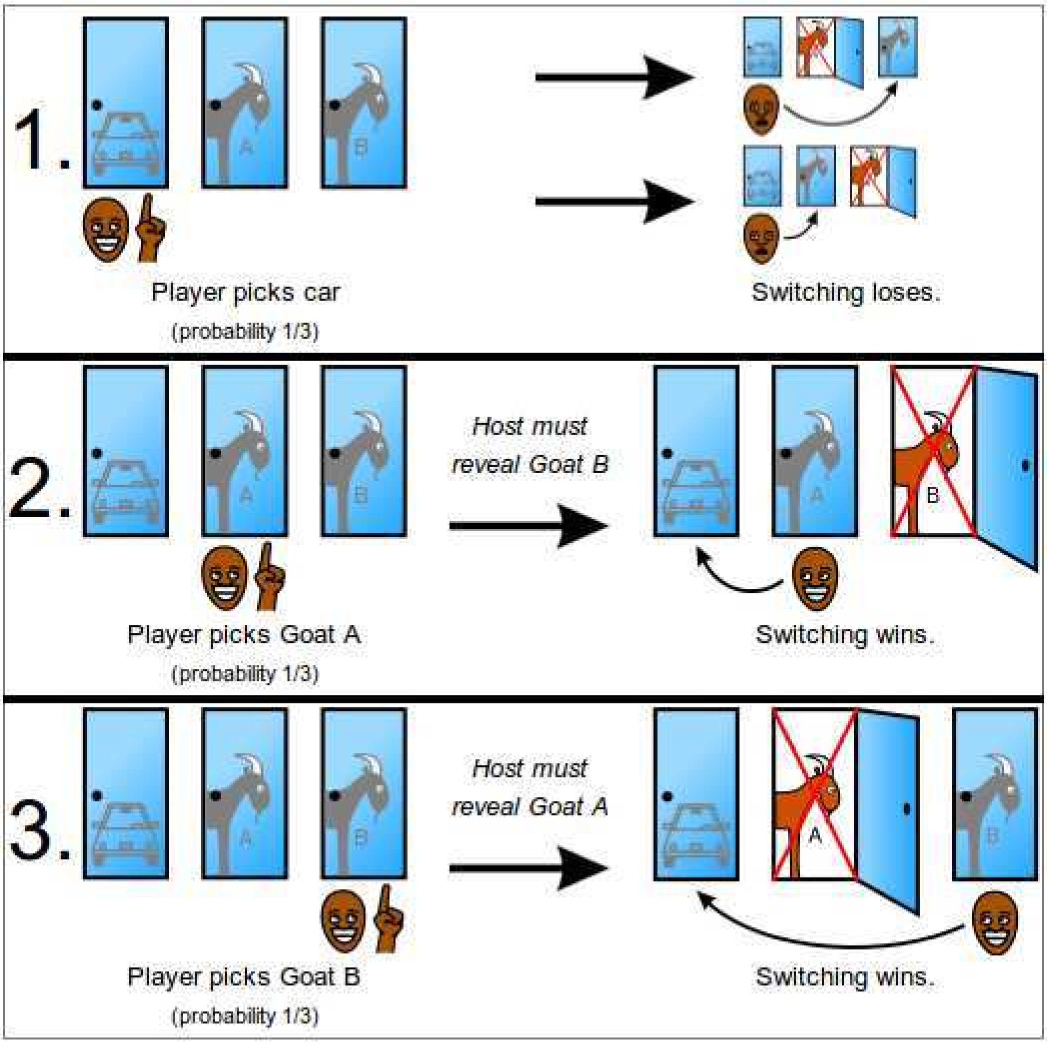
# Monty Hall problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Yes it is! See picture stolen from Wikipedia.

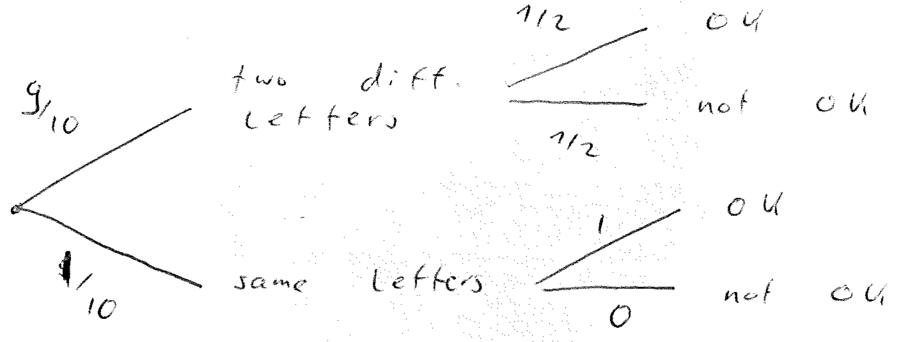


# MALMÖ

Consider the word MALMÖ.

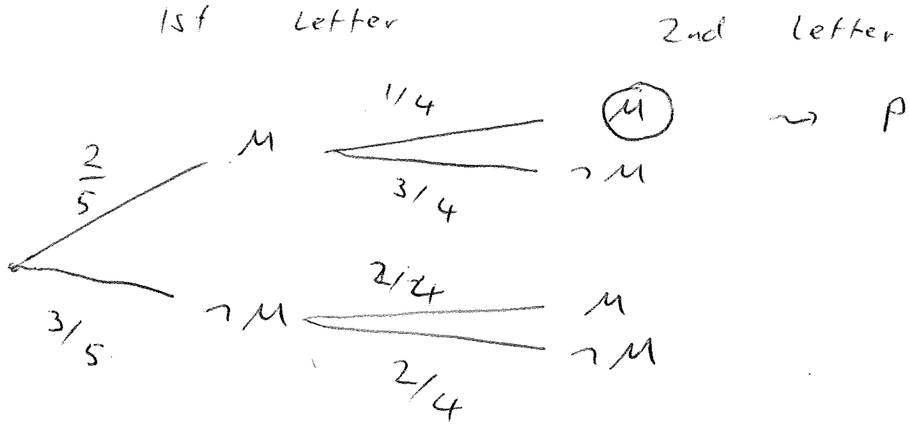
Two letters fall down. You place them back at random into the free slots.

What is prob. that the word is correct?




$$P(ok) = \frac{9}{10} \cdot \frac{1}{2} + \frac{1}{10} \cdot 1 = \frac{11}{20} \quad //$$

For the first step:




$\rightsquigarrow$ P(same) $= \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$ //

# HIV - Test

There is a test with the following properties: if you have HIV ⇝ test is positive with prob. 98%. If you don't have HIV ⇝ test is negative [η] with prob. 95%. In Sweden only 0,04% have HIV.

a) You pick a swedish person at random and do the test. What is the probability that it is positive?

b) Assume the test was positive. What is the probability that the person has HIV?

We have $\Omega = \{\text{swedish persons}\}$

$H = \{\text{swedish persons with HIV}\}$ ⇝ $H^c$ ...

$+ = \{$ -"- with positive test $\}$ ⇝ $-$ ...

$P(+ \mid H) = 0{,}98$ ; $P(- \mid H^c) = 0{,}95$ ;

$P(H) = 0{,}0004$.

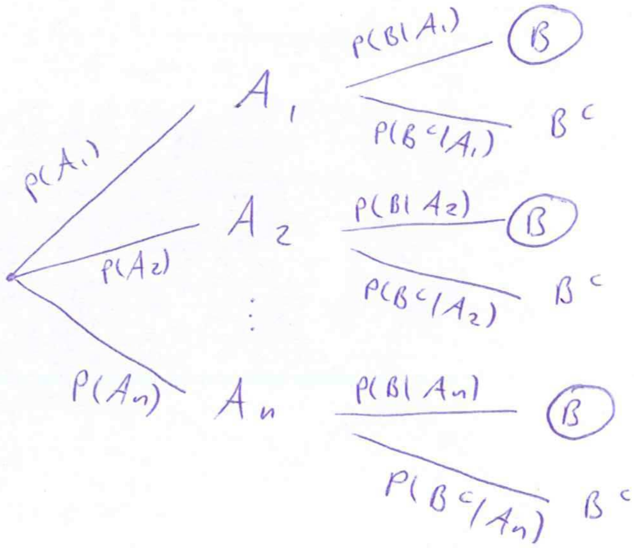
a) $P(+) = ?$ b) $P(H \mid +) = ?$

a) Law on total probability

$A_1, \dots, A_n$ disjoint with $P(A_i) > 0$

and $\bigcup A_i = \Omega$. Then

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i) \cdot P(A_i)$$




sum up!

Here: $B \mathrel{\hat{\approx}} +$ ; $A_1 \mathrel{\hat{\approx}} H$ ; $A_2 \mathrel{\hat{\approx}} H^c$

$$\Rightarrow \quad P(+) = P(+ \mid H) \cdot P(H) + P(+ \mid H^c) \cdot P(H^c)$$

$$= P(+ \mid H) \cdot P(H) + [1 - P(- \mid H^c)] \cdot [1 - P(H)]$$

$$= \dots = 0{,}050372 \mathrel{\hat{=}} 5{,}0372\,\%$$

//

b) <u>Bayes theorem</u>

$A_i$ as before. Then

$$P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(B)}$$

$$\overset{\text{total prob.}}{=} \frac{P(A_i) \cdot P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j) \cdot P(B \mid A_j)}.$$

<u>Proof:</u>

- $P(A_i \mid B) = \dfrac{P(A_i \cap B)}{P(B)} = (*)$
- $P(B \mid A_i) = \dfrac{P(B \cap A_i)}{P(A_i)} \iff P(A_i \cap B) = P(B \mid A_i) \cdot P(A_i)$

$$\Rightarrow P(A_i \mid B) = (*) = \frac{P(B \mid A_i) \cdot P(A_i)}{P(B)}. \qquad \square$$

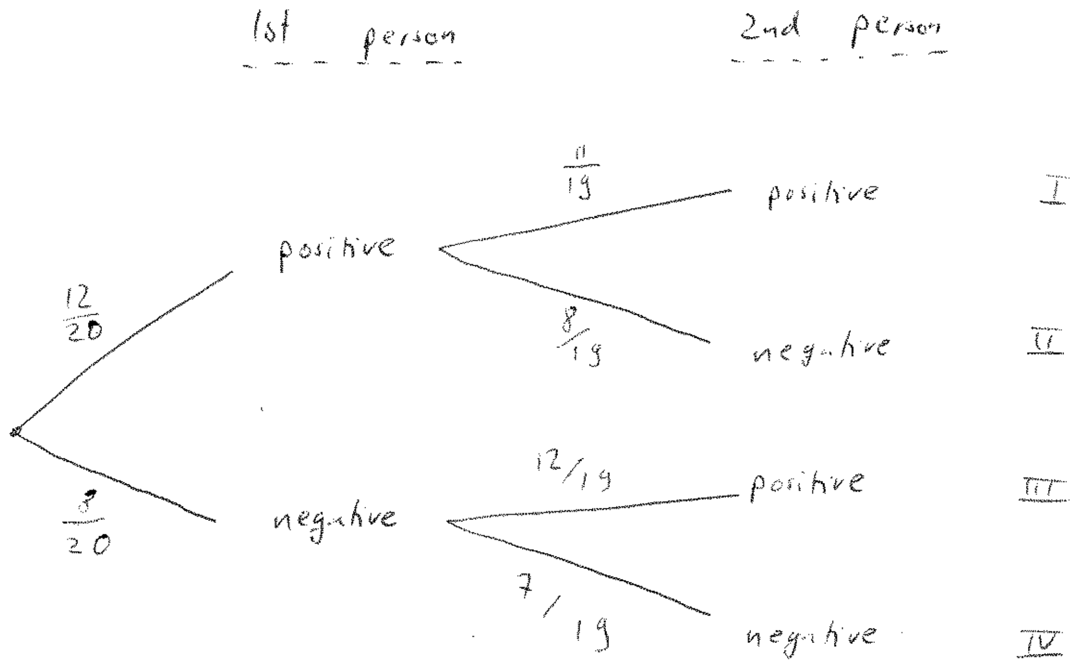
Here:

$$P(H \mid +) = \frac{P(+ \mid H) \cdot P(H)}{P(+)}$$

$$= \frac{0{,}98 \cdot 0{,}0004}{0{,}050372}$$

$$= 0{,}0078 \approx 0{,}78\,\%.$$

D.3.1




a) $$\mathbb{P}(\text{both negative}) = \mathbb{P}(\text{IV}) = \frac{8}{20} \cdot \frac{7}{19} = \frac{14}{95}$$

b) $$\mathbb{P}(\text{one positive and one negative})$$

$$= \mathbb{P}(\text{II}) + \mathbb{P}(\text{III}) = \frac{12}{20} \cdot \frac{8}{19} + \frac{8}{20} \cdot \frac{12}{19}$$

$$= \frac{192}{380} = \frac{48}{95}$$

Alternative (more formal):

$A_i$ ... the $i$th person is negative

$B_i$ ... the $i$-th person is positive

we know $P(A_1) = \frac{8}{20}$ ; $P(B_1) = \frac{12}{20}$

$P(A_2 | A_1) = \frac{7}{19}$ ; $P(B_2 | A_1) = \frac{12}{19}$

$P(A_2 | B_1) = \frac{8}{19}$ ; $P(B_2 | B_1) = \frac{11}{19}$

a) $P(\text{both negative}) = P(A_1 \cap A_2)$

$= P(A_1) \cdot P(A_2 | A_1)$

$= \frac{8}{20} \cdot \frac{7}{19} = \frac{14}{95}$ //

b) $P(\text{one positive and one negative})$

$= P((A_1 \cap B_2) \cup (A_2 \cap B_1))$

disjoint

$= P(A_1 \cap B_2) + P(A_2 \cap B_1)$

$= P(A_1) \cdot P(B_2 | A_1) + P(B_1) \cdot P(A_2 | B_1)$

$= \frac{8}{20} \cdot \frac{12}{19} + \frac{12}{20} \cdot \frac{8}{19} = \frac{24}{95}$ //

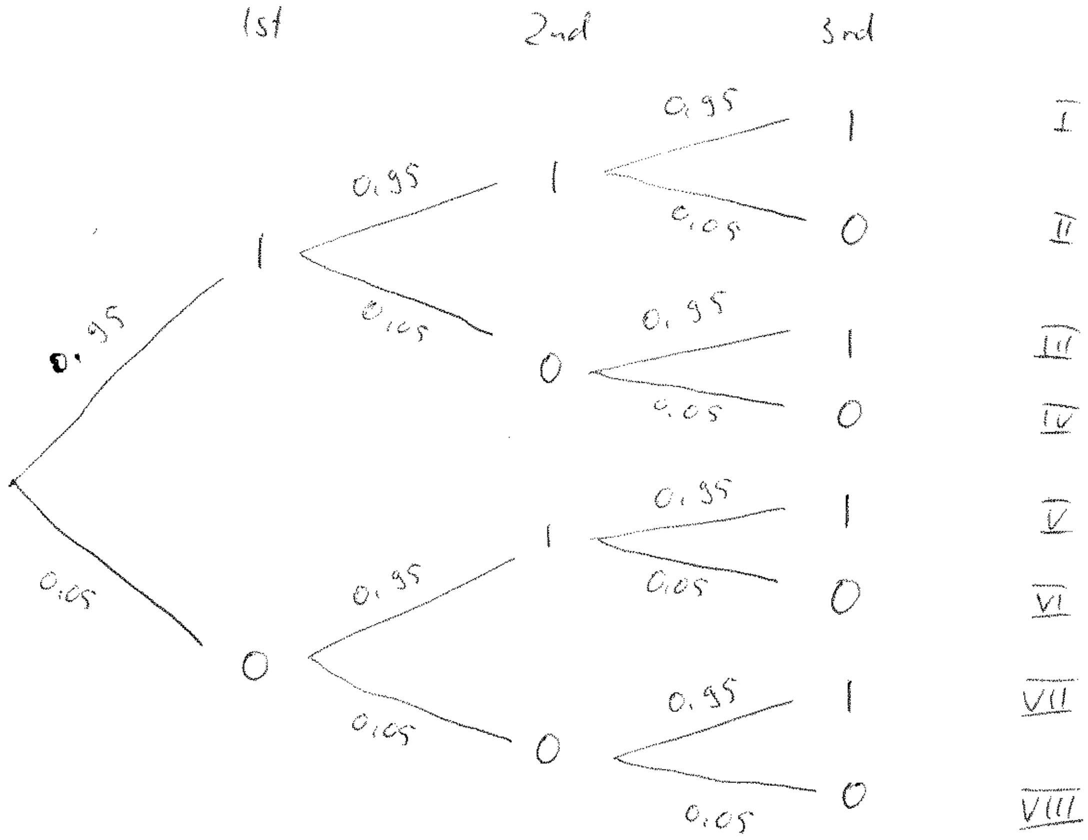
D.3.2

a) 1st system:

$\mathbb{P}$( Input 1 gives output 1 ) $= 0,95$ //

2nd system:

We send a 1 three times and check the output.




$\mathbb{P}$( Input 1 gives output 1)

$= \mathbb{P}(I) + \mathbb{P}(II) + \mathbb{P}(III) + \mathbb{P}(V)$

$= 0,95^3 + 0,95^2\, 0,05 + 0,95^2\, 0,05 + 0,95^2 \cdot 0,05$

$= 0,99275$ //

Alternative:

We send a 1 three times.

$X$ is number of received ones.

Then $X \sim Bin(3, 0.95)$.

$\mathbb{P}$( Input 1 gives output 1) $= \mathbb{P}(X \geq 2)$

$= \mathbb{P}(X = 2) + \mathbb{P}(X = 3)$

$= \underbrace{\binom{3}{2}}_{=3} \cdot 0.95^2 \cdot 0.05 + \underbrace{\binom{3}{3}}_{=1} \cdot 0.95^3$

$= 0,99275$ //

c) The second system is safer but takes more time and is probably more expensive.

b) P( correct word with 1st system)

$= 0,95^3 \approx 0,857$ //

P( correct word with 2nd system )

$= 0,99275^3 \approx 0,978$ //