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Tensor Products of Gaussian Processes

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Abstract

In the present work we introduce the tensor product of Gaussian processes: two Gaussian processes X and Y are combined by the multiplication of their covariance functions and the resulting process $X \otimes Y$ is a Gaussian process as well. For example, the fractional Brownian sheet evolves by generating the tensor product of fractional Brownian motions. The main theorem of this thesis is that the path continuity of X and Y implies the path continuity of $X \otimes Y$.

In order to prove this assertion, we introduce tensor products of Banach spaces and Hilbert spaces. Afterwards, we verify the squared integrability of the norm of Gaussian random variables by an application of Fernique's Theorem and show a result from Chevet that makes an assertion about the convergence of random series of the form $\sum_{i,j=1}^{\infty} \xi_{ij} x_i \otimes y_j$ in the injective tensor product of two Banach spaces, where the ξ_{ij}'s are independent standard normal random variables. Moreover, we develop a relation between Hilbert spaces and Gaussian random variables with values in a Banach space E. Thereby, a Gaussian process is generated out of the structure of a suitable Hilbert space H by an operator $u : H \to E$.

Finally, we show the main theorem by an appropriate application of the mentioned results.

Contents

Chapter 1

Introduction

Time-dependent random behavior like stock prices, motion of particles or the concentration of certain substances in the air is normally described by stochastic processes. The most important class of stochastic processes is the set commonly known as Gaussian processes.

Definition 1.1. A random vector η with values in $\mathbb{R}^n$ is called Gaussian if it is multidimensional normal distributed with mean zero, i.e. its characteristic function $\widehat{\eta} : \mathbb{R}^n \to \mathbb{C}$ is given by

$$\widehat{\eta}(t) = \exp(-\frac{1}{2}\langle Rt, t \rangle)$$

with $t \in \mathbb{R}^n$, where $R \in \mathbb{R}^{n\times n}$ is a symmetric and nonnegative definite matrix.

We point out explicitly that a Gaussian vector is always centered. We restrict it, because we only consider centered processes and centered random variables in this work. Now, we define Gaussian processes.

Definition 1.2. Let S be a nonempty (but otherwise arbitrary) set and $X = (X_s)_{s\in S} = (X(s))_{s\in S}$ a family of random variables, defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. X is called a Gaussian process if the vectors $(X_{s_1}, ..., X_{s_n})$ are Gaussian for every natural number n and $s_1, ..., s_n \in S$.

A second process $X' = (X'_s)_{s\in S}$ defined on a second probability space $(\Omega', \mathfrak{A}', \mathbb{P}')$, is said to be a version of X, if the distributions of X and X' coincide, i.e. the vectors $(X_{s_1}, ..., X_{s_n})$ and $(X'_{s_1}, ..., X'_{s_n})$ are equally distributed for every $n \in \mathbb{N}$ and $s_1, ..., s_n \in S$.

For Gaussian processes the covariance function is of vital importance.

Definition 1.3. The covariance function $R_X : S \times S \to \mathbb{R}$ of a Gaussian process $X = (X_s)_{s\in S}$ is defined as

$$R_X(s_1, s_2) := \mathbb{E}\, X_{s_1} X_{s_2}$$

for all $s_1, s_2 \in S$.

A mapping $R : S\times S \to \mathbb{R}$ is said to be symmetric if we have $R(s_1, s_2) = R(s_2, s_1)$ for every $s_1, s_2 \in S$ and it is said to be nonnegative definite if for every natural number n, $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and $s_1, ..., s_n \in S$, we have

$$\sum_{i,j=1}^{n} \lambda_i\lambda_j R(s_i, s_j) \geq 0.$$

It is easy to see that the covariance function of a Gaussian process $X = (X_s)_{s\in S}$ is symmetric and nonnegative definite: the symmetry is obvious and for $n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and $s_1, ..., s_n \in S$, we get

$$\sum_{i,j=1}^{n} \lambda_i\lambda_j R_X(s_i, s_j) = \mathbb{E}\,(\sum_{i,j=1}^{n} \lambda_i\lambda_j X_{s_i}X_{s_j}) = \mathbb{E}\,(\sum_{i=1}^{n} \lambda_i X_{s_i})^2 \geq 0.$$

On the other hand, we have the following

Theorem 1.1. *Let $R : S\times S \to \mathbb{R}$ be a symmetric and nonnegative definite function. Then, there is a (except for versions) unique determined Gaussian process $X = (X_s)_{s\in S}$ with $R(s_1, s_2) = \mathbb{E}\, X_{s_1}X_{s_2}$ for all $s_1, s_2 \in S$.*

Proof. This Theorem is a simple conclusion of Kolmogorov's extension Theorem (cf. Theorem 2.1.5 in [11]). □

As an example we introduce the fractional Brownian motion. For every $0 < H < 1$, the mapping $R : [0,\infty) \times [0,\infty) \to \mathbb{R}$ given by

$$R(s_1, s_2) := \frac{1}{2}(|s_1|^{2H} + |s_2|^{2H} - |s_1 - s_2|^{2H})$$

with $s_1, s_2 \in [0,\infty)$ is symmetric and nonnegative definite (cf. Lemma 2.10.8 in [13]). The unique Gaussian process with covariance function R on index set $[0,\infty)$ is called the fractional Brownian motion $B_H = (B_H(s))_{s\in[0,\infty)}$ with Hurst index H. In the case $H = 1/2$, we obtain the well-known standard Brownian motion.

One important property of the fractional Brownian motion, which makes this process very useful for applications, is the self-similarity, i.e. the finite dimensional distributions of $(B_H(cs))_{s\in[0,\infty)}$ and $(c^H B_H(s))_{s\in[0,\infty)}$ are the same for every $c > 0$. This fact can easily be verified by a comparison of the covariance functions. Figuratively speaking, this corresponds to the fact, that all zoomed pictures of this process look the same.

A second very important property of the fractional Brownian motion is the path continuity. Almost all phenomena in nature are continuous. Hence, this property is very desirable for processes that shall describe natural behavior.

Definition 1.4. Let $X = (X_s)_{s\in S}$ be a Gaussian process. For a fixed $\omega \in \Omega$ the mapping $s \mapsto X_s(\omega)$ is called a path or a trajectory of X.

If S is a topological space, X is said to be continuous almost surely (a.s.) if the paths of X are continuous for almost all $\omega \in \Omega$.

For a proof of the path continuity of the fractional Brownian motion, we refer to Proposition 3.1 in [14].

Many Gaussian processes arise by combining easier Gaussian processes. For example, one may multiply the covariance functions of two processes:

Definition 1.5. Let $X = (X_s)_{s\in S}$ and $Y = (Y_t)_{t\in T}$ be some Gaussian processes with index sets S and T, respectively. Moreover, let $R_X : S\times S \to \mathbb{R}$ and $R_Y : T\times T \to \mathbb{R}$ be the covariance functions of X and Y. We define a mapping $R : (S\times T)\times(S\times T) \to \mathbb{R}$ as

$$R((s_1, t_1), (s_2, t_2)) := R_X(s_1, s_2)R_Y(t_1, t_2)$$

for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$. Now, we denote the Gaussian process with index set $S \times T$ and covariance function R as the tensor product process $X \otimes Y$.

The existence and uniqueness of such a process is not yet clear. To see this, we need the following

Proposition 1.1. *For arbitrary nonempty sets S and T, let $R_1 : S \times S \to \mathbb{R}$ and $R_2 : T \times T \to \mathbb{R}$ be nonnegative definite mappings. Then the mapping $R : (S \times T) \times (S \times T) \to \mathbb{R}$ defined by*

$$R((s_1, t_1), (s_2, t_2)) := R_1(s_1, s_2)R_2(t_1, t_2)$$

for $s_1, s_2 \in S$ and $t_1, t_2 \in T$ is nonnegative definite as well.

Proof. We need to show that for every $n \in N$, $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and $(s_1, t_1), ..., (s_n, t_n) \in S \times T$ holds

$$\sum_{i,j=1}^{n} \lambda_i\lambda_j R((s_i, t_i), (s_j, t_j)) \geq 0.$$

By the nonnegative definiteness of R_1 and R_2, we know that the matrices

$$M_1 := \begin{pmatrix} R_1(s_1, s_1) & R_1(s_1, s_2) & \dots & R_1(s_1, s_n) \\ R_1(s_2, s_1) & R_1(s_2, s_2) & \dots & R_1(s_2, s_n) \\ \vdots & \vdots & \ddots & \vdots \\ R_1(s_n, s_1) & R_1(s_n, s_2) & \dots & R_1(s_n, s_n) \end{pmatrix} \quad \text{and}$$

$$M_2 := \begin{pmatrix} R_2(t_1, t_1) & R_2(t_1, t_2) & \dots & R_2(t_1, t_n) \\ R_2(t_2, t_1) & R_2(t_2, t_2) & \dots & R_2(t_2, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ R_2(t_n, t_1) & R_2(t_n, t_2) & \dots & R_2(t_n, t_n) \end{pmatrix}$$

are nonnegative definite and thus, the matrix

$$M := \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

is nonnegative definite as well. Hence, there is a Gaussian vector $Z = (Z_1, ..., Z_{2n})$ in $\mathbb{R}^{2n}$ with covariance structure M, i.e. $\mathbb{E}\, Z_i Z_j = R_1(s_i, s_j)$ and $\mathbb{E}\, Z_{n+i} Z_{n+j} = R_2(t_i, t_j)$ for $i, j = 1, ..., n$. Moreover, the vectors $(Z_1, ..., Z_n)$ and $(Z_{n+1}, ..., Z_{2n})$ are independent. This leads for all $\lambda_1, ..., \lambda_n \in \mathbb{R}$ to

$$\begin{aligned}\sum_{i,j=1}^{n} \lambda_i \lambda_j R((s_i, t_i), (s_j, t_j)) &= \sum_{i,j=1}^{n} \lambda_i \lambda_j \mathbb{E}\, Z_i Z_j \mathbb{E}\, Z_{n+i} Z_{n+j} \\ &= \mathbb{E} \left(\sum_{i,j=1}^{n} \lambda_i \lambda_j Z_i Z_j Z_{n+i} Z_{n+j} \right) \\ &= \mathbb{E} \left(\sum_{i=1}^{n} \lambda_i Z_i Z_{n+i} \right)^2 \geq 0. \qquad \square\end{aligned}$$

Theorem 1.2. *Let $X = (X_s)_{s \in S}$ and $Y = (Y_t)_{t \in T}$ be some Gaussian processes with index sets S and T, respectively. Then the tensor product process $X \otimes Y$ with index set $S \times T$ defined as above exists and is (except for versions) uniquely determined.*

Proof. Again, let $R_X : S \times S \to \mathbb{R}$ and $R_Y : T \times T \to \mathbb{R}$ be the covariance functions of X and Y. By Theorem 1.1 it is enough to show that $R : (S \times T) \times (S \times T) \to \mathbb{R}$ with $(s_1, t_1), (s_2, t_2) \mapsto R_X(s_1, s_2) R_Y(t_1, t_2)$ is symmetric and nonnegative definite. We know that the covariance functions R_X and R_Y are symmetric and nonnegative definite. Thus, the symmetry of R is very simple: for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$, we have

$$\begin{aligned}R((s_1, t_1), (s_2, t_2)) &= R_X(s_1, s_2) R_Y(t_1, t_2) \\ &= R_X(s_2, s_1) R_Y(t_2, t_1) \\ &= R((s_2, t_2), (s_1, t_1)).\end{aligned}$$

The nonnegative definiteness of R follows by Proposition 1.1.

Hence, we are able to apply Theorem 1.1 and get a (except for versions) unique Gaussian process Z, which is by definition the tensor product process $X \otimes Y$. $\square$

We give an example: let $0 < H_1, ..., H_d < 1$ be real numbers. The Gaussian process

$$B_H := B_{H_1} \otimes \cdots \otimes B_{H_d}$$

is called the fractional Brownian sheet in d dimensions with Hurst vector $H = (H_1, ..., H_d)$. By definition, the fractional Brownian sheet is a Gaussian process on index set $[0, \infty)^d$ with covariance function

$$\mathbb{E}\, B_H(s_1, ..., s_d) B_H(t_1, ..., t_d) = 2^{-d} \prod_{i=1}^{d} (|s_i|^{2H_i} + |t_i|^{2H_i} - |s_i - t_i|^{2H_i}).$$

The following question arises: if the processes X and Y have a certain property, is this property fulfilled by the tensor product process $X \otimes Y$ as well? For example, one may ask if the fractional Brownian sheet is - as the tensor product of the self-similar and continuous fractional Brownian motion - self-similar and continuous as well. In the present work we investigate the continuity for those processes that are indexed by compact metric spaces.

Therefore, we introduce tensor products of Banach spaces and Hilbert spaces in Chapter 2, prove results from Fernique and Chevet in Chapter 3 and 4, and find connections between Gaussian processes and Hilbert spaces in Chapter 5. Finally, in Chapter 6, we put the pieces together and obtain a proof for the following

Theorem. *Let* (S, ρ_1) *and* (T, ρ_2) *be two compact metric spaces. Then* $S \times T$ *becomes a metric space with respect to* $\rho((s_1, t_1), (s_2, t_2)) := \rho_1(s_1, s_2) + \rho_2(t_1, t_2)$. *Now, let* $X = (X_s)_{s \in S}$ *and* $Y = (Y_t)_{t \in T}$ *be two Gaussian processes with index sets* S *and* T*, respectively. Moreover, we suppose that* X *and* Y *are not identical zero. Then the paths of* $X \otimes Y$ *are a.s. continuous if and only if the paths of* X *and* Y *are a.s. continuous as well.*

Chapter 2

Tensor Products and Tensor Norms

For an investigation of tensor products of Gaussian processes we need tensor products of Banach spaces, Hilbert spaces and operators, as well as a few properties of these objects.

2.1 The Algebraic Tensor Product

Let E, F and G be real Banach spaces. A bilinear mapping $B : E \times F \to G$ is called bounded if there is a real $c > 0$ with

$$\|B(x, y)\| \leq c\|x\|\|y\|$$

for every $x \in E$ and $y \in F$. We denote the set of all bilinear and bounded mappings from $E \times F$ to G as $\mathcal{B}(E \times F, G)$. In the case $G = \mathbb{R}$ we only write $\mathcal{B}(E \times F)$. The set of all linear and bounded operators from E into F is denoted as $\mathfrak{L}(E, F)$ and in the case $F = \mathbb{R}$ simply E^*. Now, we introduce the tensor product $E \otimes F$ as a subset of $\mathcal{B}(E^* \times F^*)$.

Definition 2.1. For $x \in E$ and $y \in F$, we define an elementary tensor $x \otimes y \in \mathcal{B}(E^* \times F^*)$ as $(x \otimes y)((\varphi, \psi)) := \varphi(x)\psi(y)$ for $\varphi \in E^*$ and $\psi \in F^*$. The span of those elementary tensors is denoted as $E \otimes F$, i.e.

$$E \otimes F := \{\sum_{i=1}^{n} \lambda_i x_i \otimes y_i \mid n \in \mathbb{N}, \lambda_i \in \mathbb{R}, x_i \in E, y_i \in F\}$$

and we say $E \otimes F$ is the tensor product of E and F.

Note that the representation of an $u \in E \otimes F$ is never uniquely determined. As a very simple example, we can identify $\mathbb{R} \otimes \mathbb{R}$ and $\mathbb{R}$. The function $J : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ with $x \otimes y \mapsto xy$ for all $x, y \in \mathbb{R}$ yields an isomorphism.

Proposition 2.1. *The mapping $\otimes : E \times F \to E \otimes F$ with $(x, y) \mapsto x \otimes y$ is bilinear and for every $x \in E$ and $y \in F$, we have*

$$x \otimes 0 = 0 \otimes y = 0.$$

Proof. For every $x_1, x_2 \in E$, $y \in F$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\varphi \in E^*$ and $\psi \in E^*$, we get

$$\begin{aligned} ((\alpha_1 x_1 + \alpha_2 x_2) \otimes y)((\varphi, \psi)) &= \varphi(\alpha_1 x_1 + \alpha_2 x_2)\psi(y) \\ &= \alpha_1 \varphi(x_1)\psi(y) + \alpha_2 \varphi(x_2)\psi(y) \\ &= \alpha_1 (x_1 \otimes y)((\varphi, \psi)) + \alpha_2 (x_2 \otimes y)((\varphi, \psi)) \\ &= (\alpha_1 (x_1 \otimes y) + \alpha_2 (x_2 \otimes y))((\varphi, \psi)). \end{aligned}$$

Thus, the mapping $\otimes$ is linear in the first variable. The linearity in the second variable is shown in an analogous manner. The second part of the Proposition is obvious. □

The bilinearity of the mapping $\otimes : E \times F \to E \otimes F$ yields for every $u \in E \otimes F$ a representation of the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

with $n \in \mathbb{N}$, $x_i \in E$ and $y_i \in F$.

Proposition 2.2. *Let $E_0 \subset E$ and $F_0 \subset F$ be subsets of E and F, respectively, that are each linear independent. Then the set $\{x \otimes y \mid x \in E_0, y \in F_0\}$ is linear independent in $E \otimes F$.*

If E_0 and F_0 are bases of E and F, then $\{x \otimes y \mid x \in E_0, y \in F_0\}$ is a basis of $E \otimes F$.

Proof. For $u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i = 0$ with $x_i \in E_0$, $y_i \in F_0$ and $\lambda_i \in \mathbb{R}$, we have to show $\lambda_i = 0$ for every $i = 1, ..., n$. By the assumption $u = 0$, we obtain for every $\varphi \in E^*$ and $\psi \in F^*$

$$0 = u((\varphi, \psi)) = \sum_{i=1}^{n} \lambda_i \varphi(x_i)\psi(y_i) = \varphi(\sum_{i=1}^{n} \lambda_i x_i \psi(y_i)).$$

By Hahn Banach's Theorem (cf. Corollary III.6.7 in [4]), we get $\sum_{i=1}^{n} \lambda_i x_i \psi(y_i) = 0$ and since the x_i's are linear independent, we have $0 = \lambda_i \psi(y_i) = \psi(\lambda_i y_i)$ for every $1 \leq i \leq n$. Using Hahn Banach's Theorem a second time, we get $\lambda_i y_i = 0$ and thus, $\lambda_i = 0$ for $1 \leq i \leq n$ because the y_i's are elements of a linear independent set and so not zero.

The second part follows by the first part and the fact that the elementary tensors $x_0 \otimes y_0 \in E \otimes F$ with arbitrary $x_0 \in E$ and $y_0 \in F$ are elements of $\operatorname{span}\{x \otimes y \mid x \in E_0, y \in F_0\}$: there are representations $x_0 = \sum_{i=1}^{m} \lambda_i x_i$ and $y_0 = \sum_{j=1}^{n} \mu_j y_j$ with $\lambda_i, \mu_i \in \mathbb{R}$, $x_i \in E_0$ and $y_i \in F_0$ because we assume E_0 and F_0 to be bases of E and

F. Then we have

$$x_0 \otimes y_0 = (\sum_{i=1}^{m} \lambda_i x_i) \otimes (\sum_{j=1}^{n} \mu_j y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j x_i \otimes y_j. \qquad \square$$

Proposition 2.3. *For every $u \in E \otimes F$, there is a smallest natural number n so that there is a representation $u = \sum_{i=1}^{n} x_i \otimes y_i$ with $x_i \in E$ and $y_i \in F$. Then the sets $\{x_1, ..., x_n\} \subset E$ and $\{y_1, ..., y_n\} \subset F$ are each linear independent.*

Proof. Let $\sum_{i=1}^{n} x_i \otimes y_i$ be a minimal representation of $u \in E \otimes F$. We show the linear independence of $\{x_1, ..., x_n\}$. The independence of $\{y_1, ..., y_n\}$ can be seen in an analogous manner. We assume that $\{x_1, ..., x_n\}$ is not linear independent. Without loss of generality, x_n may be representable by a linear combination of $x_1, ..., x_{n-1}$, i.e. $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$ with suitable $\alpha_i \in \mathbb{R}$. Thus, we have

$$\begin{aligned} u &= \sum_{i=1}^{n-1} x_i \otimes y_i + (\sum_{i=1}^{n-1} \alpha_i x_i) \otimes y_n \\ &= \sum_{i=1}^{n-1} (x_i \otimes y_i + \alpha_i x_i \otimes y_n) \\ &= \sum_{i=1}^{n-1} x_i \otimes (y_i + \alpha_i y_n). \end{aligned}$$

Setting $y_i' = y_i + \alpha_i y_n$ for $1 \leq i \leq n-1$, we get by $\sum_{i=1}^{n-1} x_i \otimes y_i'$ a representation of u with $n-1$ summands, which is a contradiction to the assumption that n is minimal. Hence, the set $\{x_1, ..., x_n\} \subset E$ must be linear independent. $\square$

Let E be a Banach space. A subset $G \subset E^*$ is called separating if there is for every $x, y \in E$ with $x \neq y$ a functional $\varphi \in G$ with $\varphi(x) \neq \varphi(y)$.

Proposition 2.4. *For $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$, the following statements are equivalent.*

i. $u = 0$, *i.e.* $u((\varphi, \psi)) = \sum_{i=1}^{n} \varphi(x_i)\psi(y_i) = 0$ *for all* $\varphi \in E^*$ *and* $\psi \in F^*$.

ii. $\sum_{i=1}^{n} \varphi(x_i) y_i = 0$ *for all* $\varphi \in E^*$.

iii. $\sum_{i=1}^{n} x_i \psi(y_i) = 0$ *for all* $\psi \in F^*$.

iv. $\sum_{i=1}^{n} \varphi(x_i) y_i = 0$ *for all* $\varphi \in G$ *where* G *is a separating subset of* E^*.

v. $\sum_{i=1}^{n} x_i \psi(y_i) = 0$ *for all* $\psi \in H$ *where* H *is a separating subset of* F^*.

Proof. The steps $(i) \Rightarrow (ii) \Rightarrow (iv)$ and $(i) \Rightarrow (iii) \Rightarrow (v)$ are easy consequences of the Hahn Banach Theorem. So we only show the step $(iv) \Rightarrow (i)$. The step $(v) \Rightarrow (i)$ can be seen analogously. With the assumption, we have

$$0 = \psi(\sum_{i=1}^{n} \varphi(x_i) y_i) = \sum_{i=1}^{n} \varphi(x_i)\psi(y_i) = \varphi(\sum_{i=1}^{n} x_i \psi(y_i))$$

for every $\varphi \in G$ and $\psi \in F^*$. Hence, since G is separating, we get $\sum_{i=1}^{n} x_i \psi(y_i) = 0$ and thus, for every $\varphi \in E^*$ and $\psi \in F^*$

$$\sum_{i=1}^{n} \varphi(x_i)\psi(y_i) = \varphi(\sum_{i=1}^{n} x_i \psi(y_i)) = 0. \qquad \square$$

Definition 2.2. Let E, F, G and H be four Banach spaces and let $a : E \to G$ and $b : F \to H$ be linear operators. Then the linear operator $c : E \otimes F \to G \otimes H$ with

$$c(x \otimes y) = a(x) \otimes b(y)$$

for all $x \in E$ and $y \in F$ is called the tensor product operator $a \otimes b$.

Note that we can not do any assertions about the boundedness of $a \otimes b$ yet, because until now there are no norms for the tensor product spaces.

2.2 The Injective Tensor Norm

In a Banach space E the abbreviation $B_\varepsilon(x)$ denotes the closed ball of radius $\varepsilon > 0$ and centered at $x \in E$. We also define $B_E := B_1(0)$ and say B_E is the closed unit ball of E. A subset $G \subset B_{E^*}$ is said to be a norming subset if we have $\|x\| = \sup_{\varphi \in G} |\varphi(x)|$ for every $x \in E$.

Proposition 2.5. *The mapping* $\epsilon : X \otimes Y \to \mathbb{R}$ *with*

$$\epsilon(u) := \sup \left\{ \left| \sum_{i=1}^{n} \varphi(x_i)\psi(y_i) \right| : \varphi \in B_{E^*}, \psi \in B_{F^*} \right\}$$

for $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y \subset \mathcal{B}(E^* \times F^*)$ *defines a norm on* $X \otimes Y$.

Proof. Firstly, ϵ is well defined because two elements $u_1 = \sum_{i=1}^{n_1} x_i^{(1)} \otimes y_i^{(1)} \in X \otimes Y$ and $u_2 = \sum_{i=1}^{n_2} x_i^{(2)} \otimes y_i^{(2)} \in X \otimes Y$ are equal if we have

$$\sum_{i=1}^{n_1} \varphi(x_i^{(1)})\psi(y_i^{(1)}) = \sum_{i=1}^{n_2} \varphi(x_i^{(2)})\psi(y_i^{(2)})$$

for all $\varphi \in E^*$ and $\psi \in F^*$. By the definition of the tensor product follows $\epsilon(u) = 0$ if and only if $u = 0$ for $u \in X \otimes Y$. For a real λ and $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, we have

$$\begin{aligned}\epsilon(\lambda u) &= \epsilon(\sum_{i=1}^{n} (\lambda x_i) \otimes y_i) = \sup \left\{ \left| \sum_{i=1}^{n} \varphi(\lambda x_i)\psi(y_i) \right| : \varphi \in B_{E^*}, \psi \in B_{F^*} \right\} \\ &= \sup \left\{ |\lambda| \left| \sum_{i=1}^{n} \varphi(x_i)\psi(y_i) \right| : \varphi \in B_{E^*}, \psi \in B_{F^*} \right\} = |\lambda|\epsilon(u).\end{aligned}$$

Finally, the triangle inequality can be seen as follows: for $u_1 = \sum_{i=1}^{n_1} x_i^{(1)} \otimes y_i^{(1)}$ and $u_2 = \sum_{i=1}^{n_2} x_i^{(2)} \otimes y_i^{(2)}$, we get

$$\begin{aligned}\epsilon(u_1+u_2) &= \sup\left\{\left|\sum_{i=1}^{n_1} \varphi(x_i^{(1)})\psi(y_i^{(1)}) + \sum_{i=1}^{n_2} \varphi(x_i^{(2)})\psi(y_i^{(2)})\right| : \varphi \in B_{E^*}, \psi \in B_{F^*}\right\} \\ &\leq \sup\left\{\left|\sum_{i=1}^{n_1} \varphi(x_i^{(1)})\psi(y_i^{(1)})\right| + \left|\sum_{i=1}^{n_2} \varphi(x_i^{(2)})\psi(y_i^{(2)})\right| : \varphi \in B_{E^*}, \psi \in B_{F^*}\right\}.\end{aligned}$$

Now we split the supremum into two ones and obtain

$$\begin{aligned}\epsilon(u_1+u_2) \leq \sup&\left\{\left|\sum_{i=1}^{n_1} \varphi(x_i^{(1)})\psi(y_i^{(1)})\right| : \varphi \in B_{E^*}, \psi \in B_{F^*}\right\} \\ &+ \sup\left\{\left|\sum_{i=1}^{n_2} \varphi(x_i^{(2)})\psi(y_i^{(2)})\right| : \varphi \in B_{E^*}, \psi \in B_{F^*}\right\} \\ =\ &\epsilon(u_1) + \epsilon(u_2).\end{aligned}$$

□

Definition 2.3. We denote by $E \otimes_\epsilon F$ the tensor product of E and F furnished with the injective norm ϵ as defined above and its closure in $\mathcal{B}(E^* \times F^*)$ as $E \widehat{\otimes}_\epsilon F$.

Note that it is enough to take the supremum over norming subsets of B_{E^*} and B_{F^*}. Moreover, it is easy to see that $\epsilon(x \otimes y) = \|x\|\|y\|$ for every $x \in E$ and $y \in F$.

Proposition 2.6. *Let E, F, G and H be four Banach spaces and let $a \in \mathfrak{L}(E, G)$ and $b \in \mathfrak{L}(F, H)$ be linear and bounded operators. Then the linear operator $a \otimes b : E \otimes F \to G \otimes H$ is bounded from $E \otimes_\epsilon F$ to $G \otimes_\epsilon H$ and we have $\|a \otimes b\| \leq \|a\|\|b\|$.*

Proof. For every $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$, we get

$$\begin{aligned}\epsilon((a\otimes b)(u)) &= \sup\left\{\left|\sum_{i=1}^{n} \varphi(a(x_i))\psi(b(y_i))\right| : \varphi \in B_{G^*}, \psi \in B_{H^*}\right\} \\ &= \sup\left\{\left|\sum_{i=1}^{n} (a^*\varphi)(x_i)(b^*\psi)(y_i)\right| : \varphi \in B_{G^*}, \psi \in B_{H^*}\right\} \\ &\leq \sup\left\{\left|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)\right| : \varphi \in \|a^*\|B_{E^*}, \psi \in \|b^*\|B_{F^*}\right\},\end{aligned}$$

We take $\|a^*\|$ and $\|b^*\|$ out of the supremum and obtain

$$\begin{aligned}\epsilon((a\otimes b)(u)) &\leq \|a^*\|\|b^*\| \sup\left\{\left|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)\right| : \varphi \in B_{E^*}, \psi \in B_{F^*}\right\} \\ &= \|a\|\|b\|\epsilon(u)\end{aligned}$$

where we used $\|a^*\| = \|a\|$ and $\|b^*\| = \|b\|$ in the last line. Thus, $a \otimes b$ is bounded with $\|a \otimes b\| \leq \|a\|\|b\|$. □

For the unique determined extension of $a \otimes b$ to a linear and bounded operator from $E \widehat{\otimes}_\epsilon F$ to $G \widehat{\otimes}_\epsilon H$, we write $a \otimes b$, too.

Let (S, ρ) be a compact metric space and let E be a Banach space. We denote by $C(S, E)$ the space of all continuous mappings from S into E. This space becomes with respect to the norm

$$\|f\|_\infty := \sup_{s \in S} \|f(s)\|, \quad f \in C(S, E)$$

a Banach Space (cf. Proposition III.1.7 in [4]). In the case $E = \mathbb{R}$, we only write $C(S)$.

The dual space of $C(S)$ is isomorphic to the space

$$\mathcal{M}(S) := \{\lambda \mid \lambda \text{ signed finite Borel measure on S}\}.$$

An adequate pairing between $C(S)^*$ and $\mathcal{M}(S)$ is given by

$$\lambda(f) = \int_S f(s) d\lambda(s)$$

for $f \in C(S)$ and $\lambda \in \mathcal{M}(S)$ (cf. Appendix C in [4]).

For every $s \in S$, we define $\delta_s \in \mathcal{M}(S)$ by $\delta_s(f) = f(s)$ for $f \in C(S)$.

Proposition 2.7. *The set* $\mathfrak{D} := \{\delta_s \mid s \in S\}$ *is a separating and norming subset of* $\mathcal{M}(S)$. *Moreover, we have* $\|\delta_s\| = 1$ *for every* $s \in S$.

Proof. That $\mathfrak{D}$ is a norming subset is obvious by the definition of the norm on $C(S)$. Now let $f_1, f_2 \in C(S)$ be two mappings with $f_1 \neq f_2$. Hence, there is at least one $s_0 \in S$ with $\delta_{s_0}(f_1) = f_1(s_0) \neq f_2(s_0) = \delta_{s_0}(f_2)$. Thus, $\mathfrak{D}$ is also separating. Finally, we have for every $s \in S$ and $f \in C(S)$

$$|\delta_s(f)| = |f(s)| \leq \sup_{s \in S} |f(s)| = \|f\|_\infty.$$

With the continuous function $f_0 \equiv 1$ we get $|\delta_s(f_0)| = 1$ which yields $\|\delta_s\| = 1$ for every $s \in S$. □

If we have two compact metric spaces (S, ρ_1) and (T, ρ_2), the mapping $\rho : (S \times T) \times (S \times T) \to [0, \infty)$ with $\rho((s_1, t_1), (s_2, t_2)) := \rho_1(s_1, s_2) + \rho_2(t_1, t_2)$ defines a metric on $S \times T$. With this metric the product $S \times T$ is a compact metric space as well.

Theorem 2.1. *Let* (S, ρ_1) *and* (T, ρ_2) *be two compact metric spaces and let* E *be a Banach space. Then we can identify the spaces* $C(S) \widehat{\otimes}_\epsilon E$ *and* $C(S, E)$. *Moreover, we are able to identify* $C(S) \widehat{\otimes}_\epsilon C(T)$ *and* $C(S \times T)$.

Proof. In the first step we show that $J : C(S) \otimes_\epsilon E \to C(S, E)$ with $u = \sum_{i=1}^n f_i \otimes x_i \mapsto \sum_{i=1}^n f_i x_i$ yields an isometrically embedding of $C(S) \otimes_\epsilon E$ into $C(S, E)$. Firstly, we prove that J is well defined. We assume $u = \sum_{i=1}^n f_i \otimes x_i = 0 \in C(S) \otimes_\epsilon E$. This is by Proposition 2.4 equivalent to $\sum_{i=1}^n \varphi(f_i)x_i = 0$ for every $\varphi \in C(S)^*$. In particular we have

$$(Ju)(s) = \sum_{i=1}^n \delta_s(f_i)x_i = 0$$

for all $s \in S$. Hence, the mapping J is well defined.

The image of J is always a continuous function: for $u = \sum_{i=1}^n f_i \otimes x_i$ and $\varepsilon > 0$, we choose a $\delta > 0$ so that we have $|f_i(s_1) - f_i(s_2)| < \varepsilon/(n \max_j \|x_j\|)$ for all $s_1, s_2 \in S$ with $\rho(s_1, s_2) < \delta$ and for every $i = 1, ..., n$. This is possible because the f_i's are continuous. Then we obtain

$$\begin{aligned}\|(Ju)(s_1) - (Ju)(s_2)\| &= \|\sum_{i=1}^n (f_i(s_1) - f_i(s_2))x_i\| \\ &\leq \sum_{i=1}^n |f_i(s_1) - f_i(s_2)| \|x_i\| \\ &< \max_j \|x_j\| \sum_{i=1}^n \frac{\varepsilon}{n \max_j \|x_j\|} = \varepsilon\end{aligned}$$

and thus, the image Ju is a continuous mapping from S into E.

Now, let us turn to the injectivity of J: for $u = \sum_{i=1}^n f_i \otimes x_i$, we assume $Ju = 0$, i.e. $(Ju)(s) = \sum_{i=1}^n f_i(s)x_i = \sum_{i=1}^n \delta_s(f_i)x_i = 0$ for every $s \in S$. Since $\{\delta_s \mid s \in S\}$ is separating, Proposition 2.4 yields $u = 0$.

That J is isometric can be seen as follows: by Hahn Banach's Theorem, we have

$$\begin{aligned}\|Ju\|_\infty &= \sup_{s \in S} \|\sum_{i=1}^n f_i(s)x_i\| \\ &= \sup_{s \in S} \sup_{\psi \in B_{E^*}} |\psi(\sum_{i=1}^n f_i(s)x_i)| \\ &= \sup_{s \in S} \sup_{\psi \in B_{E^*}} |\sum_{i=1}^n \delta_s(f_i)\psi(x_i)|.\end{aligned}$$

We know that the set $\{\delta_s \mid s \in S\}$ is a norming subset of $\mathcal{M}(S)$. Therefore, we can write

$$\|Ju\|_\infty = \sup_{\varphi \in B_{C(S)^*}} \sup_{\psi \in B_{E^*}} |\sum_{i=1}^n \varphi(f_i)\psi(x_i)| = \epsilon(u)$$

which yields the desired.

In the second step, we show that $J(C(S) \otimes_\epsilon E)$ is dense in $C(S, E)$. Let $f \in C(S, E)$ be arbitrary and $\varepsilon > 0$. Because S is compact and the continuous image of a compact set is compact as well, we can find $s_1, ..., s_n \in S$ so that $\bigcup_{i=1}^n B_\varepsilon(f(s_i))$ covers $f(S)$. For $i = 1, ..., n$, we define open sets $V_i := \{s \in S \mid \|f(s) - f(s_i)\| < \varepsilon\}$. Then we have $S = \bigcup_{i=1}^n V_i$. With the concept of the partition of unity (see for example Theorem D.4.7 in [7]), there are continuous mappings $g_1, ..., g_n \in C(S)$ with $g_i(S) \in [0,1]$, $\text{supp}(g_i) \subset V_i$ and $\sum_{i=1}^n g_i(s) = 1$ for every $s \in S$. Now, for $u := \sum_{i=1}^n g_i \otimes f(s_i) \in C(S) \otimes_\epsilon E$, we have for every $s \in S$

$$\begin{aligned}\|(Ju - f)(s)\| &= \|\sum_{i=1}^n g_i(s) f(s_i) - f(s)\| \\ &= \|\sum_{i=1}^n g_i(s)(f(s_i) - f(s))\| \\ &\leq \sum_{i=1}^n g_i(s)\|f(s_i) - f(s)\|.\end{aligned}$$

By the construction of g_i and V_i, we have $g_i(s) = 0$ for all $i = 1, ..., n$ with $s \notin V_i$ and $\|f(s_i) - f(s)\| < \varepsilon$ for all $i = 1, ..., n$ with $s \in V_i$. This leads to

$$\|(Ju - f)(s)\| \leq \sum_{\substack{i \text{ with} \\ s \in V_i}} g_i(s)\|f(s_i) - f(s)\| < \varepsilon \sum_{\substack{i \text{ with} \\ s \in V_i}} g_i(s) \leq \varepsilon.$$

Since this is true for every $s \in S$ we get $\|Ju - f\|_\infty < \varepsilon$.

With the extension of J to $C(S) \widehat{\otimes}_\epsilon E$ we obtain an isometric isomorphism from $C(S) \widehat{\otimes}_\epsilon E$ onto $C(S, E)$, which completes the first part.

Now, if we have two compact metric spaces (S, ρ_1) and (T, ρ_2), we get with the first part $C(S) \widehat{\otimes}_\epsilon C(T) \cong C(S, C(T))$. In this step, we show $C(S \times T) \cong C(S, C(T))$ which yields the second part of the theorem. Therefore we define a mapping $J : C(S \times T) \to C(S, C(T))$ by $(Jf)(s)(t) := f(s,t)$ for all $f \in C(S \times T)$, $s \in S$ and $t \in T$.

Of course, we have to show that Jf is continuous from S to $C(T)$ for every $f \in C(S \times T)$. Since f is a continuous function on a compact set, f is uniformly continuous, i.e. in particular, there is for every $\varepsilon > 0$ a real $\delta > 0$ so that we have $|f(s_0, t) - f(s, t)| < \varepsilon$ for every $s_0, s \in S$ and $t \in T$ with $\rho((s_0, t), (s, t)) = \rho_1(s_0, s) < \delta$. Thus, there is for every fixed $s_0 \in S$ and $\varepsilon > 0$ a real $\delta > 0$ with

$$\begin{aligned}\|(Jf)(s_0) - (Jf)(s)\|_\infty &= \sup_{t \in T} |(Jf)(s_0)(t) - (Jf)(s)(t)| \\ &= \sup_{t \in T} |f(s_0, t) - f(s, t)| < \varepsilon\end{aligned}$$

for all $s \in S$ with $\rho_1(s_0, s) < \delta$.

The linearity and injectivity of J is obvious. J is also an isometry: for $f \in C(S \times T)$, we have

$$\begin{aligned}\|Jf\|_\infty &= \sup_{s \in S} \|(Jf)(s)\|_\infty = \sup_{s \in S} \sup_{t \in T} |(Jf)(s)(t)| \\ &= \sup_{(s,t) \in S \times T} |f(s,t)| = \|f\|_\infty.\end{aligned}$$

In order to show that J is also surjective, let $f_0 \in C(S, C(T))$ be fixed. We define a mapping $f : S \times T \to \mathbb{R}$ by $f(s,t) := f_0(s)(t)$ and prove that f is continuous on $S \times T$. Then we have $(Jf)(s)(t) = f(s,t) = f_0(s)(t)$ for every $s \in S$ and $t \in T$ and thus, J is surjective. To the continuity of f: let $(s_0, t_0) \in S \times T$ be fixed. Then we have for every $(s,t) \in S \times T$

$$\begin{aligned} |f(s_0,t_0) - f(s,t)| &= |f_0(s_0)(t_0) - f_0(s)(t)| \\ &\leq |f_0(s_0)(t_0) - f_0(s_0)(t)| + |f_0(s_0)(t) - f_0(s)(t)|. \end{aligned}$$

The continuous function $f_0(s_0) \in C(T)$ and the element $t_0 \in T$ are fixed. Thus, there is for $\varepsilon > 0$ a real $\delta_1 > 0$ with $|f_0(s_0)(t_0) - f_0(s_0)(t)| < \varepsilon/2$ for every $t \in T$ with $\rho_2(t_0, t) < \delta_1$. The second summand can be estimated as follows:

$$\begin{aligned} |f_0(s_0)(t) - f_0(s)(t)| &\leq \sup_{t \in T} |f_0(s_0)(t) - f_0(s)(t)| \\ &= \|f_0(s_0) - f_0(s)\|_\infty. \end{aligned}$$

The mapping f_0 is continuous from S to $C(T)$. Hence, there is a real $\delta_2 > 0$ with $\|f_0(s_0) - f_0(s)\|_\infty < \varepsilon/2$ for every $s \in S$ with $\rho_1(s_0, s) < \delta_2$ which yields $|f_0(s_0)(t) - f_0(s)(t)| < \varepsilon/2$ for every $t \in T$ and $s \in S$ with $\rho_1(s_0, s) < \delta_2$. We set $\delta := \min\{\delta_1, \delta_2\}$ and obtain

$$|f(s_0,t_0) - f(s,t)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for every $(s,t) \in S \times T$ with $\rho((s_0,t_0),(s,t)) = \rho_1(s_0,s) + \rho_2(t_0,t) < \delta$.

Therefore, the proof of the theorem is complete. □

2.3 The Tensor Product of Hilbert Spaces

In the case of two real Hilbert spaces H_1 and H_2 the injective tensor product is also well defined. However, in general $H_1 \widehat{\otimes}_\epsilon H_2$ is not a Hilbert space. To overcome this difficulty we need another approach for norming $H_1 \otimes H_2$.

For a Hilbert space H, we define a mapping $J_H : H \to H^*$ by

$$J_H(h_1)(h_2) := \langle h_1, h_2 \rangle_H$$

for all $h_1, h_2 \in H$. By the following Theorem, J_H becomes an isometric isomorphism.

Theorem 2.2 (Riesz Representation Theorem). *If $\varphi : H \to \mathbb{R}$ is a linear functional, there is a unique element $h_0 \in H$ so that $\varphi(h) = \langle h, h_0 \rangle_H$ for every $h \in H$. Moreover, we have $\|\varphi\| = \|h_0\|$.*

Proof. We refer to Theorem I.3.4 in [4]. □

Proposition 2.8. *Let $\langle \cdot, \cdot \rangle_{H_1}$ and $\langle \cdot, \cdot \rangle_{H_2}$ be the scalar products of the Hilbert spaces H_1 and H_2, respectively. Then the mapping $\langle \cdot, \cdot \rangle_2 : H_1 \otimes H_2 \times H_1 \otimes H_2 \to \mathbb{R}$ given* *by*

$$\langle u_1, u_2\rangle_2 = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\langle x_i^{(1)}, x_j^{(2)}\rangle_{H_1}\langle y_i^{(1)}, y_j^{(2)}\rangle_{H_2}$$

for all $u_1 = \sum_{i=1}^{n_1} x_i^{(1)} \otimes y_i^{(1)} \in H_1 \otimes H_2$ *and* $u_2 = \sum_{j=1}^{n_2} x_j^{(2)} \otimes y_j^{(2)} \in H_1 \otimes H_2$ *defines a scalar product on* $H_1 \otimes H_2$*.*

Proof. Firstly, we have to show that $\langle u, v\rangle_2$ is independent from the representation of $u, v \in H_1 \otimes H_2$. For this purpose, we assume that u and v can be represented by

$$\begin{aligned} u &= \sum_{i=1}^{n_1} x_i^{(1,1)} \otimes y_i^{(1,1)} = \sum_{i=1}^{m_1} x_i^{(1,2)} \otimes y_i^{(1,2)} \qquad \text{and} \\ v &= \sum_{j=1}^{n_2} x_j^{(2,1)} \otimes y_j^{(2,1)} = \sum_{j=1}^{m_2} x_j^{(2,2)} \otimes y_j^{(2,2)}. \end{aligned}$$

By definition, this is true if and only if we have for every $\varphi \in H_1^*$ and $\psi \in H_2^*$

$$\begin{aligned} \sum_{i=1}^{n_1} \varphi(x_i^{(1,1)})\psi(y_i^{(1,1)}) &= \sum_{i=1}^{m_1} \varphi(x_i^{(1,2)})\psi(y_i^{(1,2)}) \qquad \text{and} \\ \sum_{j=1}^{n_2} \varphi(x_j^{(2,1)})\psi(y_j^{(2,1)}) &= \sum_{j=1}^{m_2} \varphi(x_j^{(2,2)})\psi(y_j^{(2,2)}). \end{aligned}$$

Using the operators J_{H_1} and J_{H_2} that identify the Hilbert spaces H_1 and H_2 with their duals, we get

$$\begin{aligned} \langle\sum_{i=1}^{n_1} x_i^{(1,1)} \otimes y_i^{(1,1)}, \sum_{j=1}^{n_2} x_j^{(2,1)} \otimes y_j^{(2,1)}\rangle_2 &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\langle x_i^{(1,1)}, x_j^{(2,1)}\rangle_{H_1}\langle y_i^{(1,1)}, y_j^{(2,1)}\rangle_{H_2} \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(J_{H_1}x_i^{(1,1)})(x_j^{(2,1)})(J_{H_2}y_i^{(1,1)})(y_j^{(2,1)}). \end{aligned}$$

We apply the identity of the representations for v and obtain

$$\begin{aligned} \langle\sum_{i=1}^{n_1} x_i^{(1,1)} \otimes y_i^{(1,1)}, \sum_{j=1}^{n_2} x_j^{(2,1)} \otimes y_j^{(2,1)}\rangle_2 &= \sum_{i=1}^{n_1}\sum_{j=1}^{m_2}(J_{H_1}x_i^{(1,1)})(x_j^{(2,2)})(J_{H_2}y_i^{(1,1)})(y_j^{(2,2)}) \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{m_2}\langle x_i^{(1,1)}, x_j^{(2,2)}\rangle_{H_1}\langle y_i^{(1,1)}, y_j^{(2,2)}\rangle_{H_2}. \end{aligned}$$

In an analogous manner, the identity of the representations for u finally leads to

$$\begin{aligned}\langle\sum_{i=1}^{n_1} x_i^{(1,1)} \otimes y_i^{(1,1)}, \sum_{j=1}^{n_2} x_j^{(2,1)} \otimes y_j^{(2,1)}\rangle_2 &= \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\langle x_i^{(1,2)}, x_j^{(2,2)}\rangle_{H_1}\langle y_i^{(1,2)}, y_j^{(2,2)}\rangle_{H_2} \\ &= \langle\sum_{i=1}^{m_1} x_i^{(1,2)} \otimes y_i^{(1,2)}, \sum_{j=1}^{m_2} x_j^{(2,2)} \otimes y_j^{(2,2)}\rangle_2.\end{aligned}$$

As the product of two nonnegative definite mappings, $\langle\cdot,\cdot\rangle_2$ is by Proposition 1.1 nonnegative definite as well, i.e. for every $u = \sum_{i=1}^n x_i \otimes y_i$ with $x_i \in H_1$ and $y_i \in H_2$, we have

$$\langle u, u\rangle_2 = \sum_{i=1}^n\sum_{j=1}^n \langle x_i, x_j\rangle_{H_1}\langle y_i, y_j\rangle_{H_2} \geq 0$$

and thus, $\langle u, u\rangle_2 \geq 0$ for all $u \in H_1 \widehat{\otimes}_2 H_2$.

With the independence of $\langle u, u\rangle_2$ from the representation of $u \in H_1 \widehat{\otimes}_2 H_2$, we get $\langle u, u\rangle_2 = 0$ for $u = 0$. On the other hand $\langle u, u\rangle_2 = 0$ yields $u = 0$: therefore, let $u = \sum_{i=1}^n x_i \otimes y_i$ be a minimal representation. Then the set $\{x_1, ..., x_n\} \subset H_1$ is by Proposition 2.3 linear independent. Now, the Gram-Schmidt process (cf. Process I.4.6 [4]) yields with

$$x_i' := x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, x_j'\rangle_{H_1}}{\langle x_j', x_j'\rangle_{H_1}} x_j'$$

for $i = 1, ..., n$ an orthogonal system $\{x_1', ..., x_n'\}$. Moreover, we have

$$\begin{aligned}u &= \sum_{i=1}^n (x_i' + \sum_{j=1}^{i-1} \frac{\langle x_i, x_j'\rangle_{H_1}}{\langle x_j', x_j'\rangle_{H_1}} x_j') \otimes y_i \\ &= \sum_{i=1}^n x_i' \otimes y_i + \sum_{j=1}^{i-1} \frac{\langle x_i, x_j'\rangle_{H_1}}{\langle x_j', x_j'\rangle_{H_1}} x_j' \otimes y_i \\ &= \sum_{i=1}^n x_i' \otimes (y_i + \sum_{j=i+1}^{n} \frac{\langle x_j, x_i'\rangle_{H_1}}{\langle x_i', x_i'\rangle_{H_1}} y_j).\end{aligned}$$

By setting

$$y_i' := y_i + \sum_{j=i+1}^{n} \frac{\langle x_j, x_i'\rangle_{H_1}}{\langle x_i', x_i'\rangle_{H_1}} y_j$$

for $i = 1, ..., n$ we obtain

$$0 = \langle u, u\rangle_2 = \sum_{i=1}^n\sum_{j=1}^n \langle x_i', x_j'\rangle_{H_1}\langle y_i', y_j'\rangle_{H_2} = \sum_{i=1}^n \langle x_i', x_i'\rangle_{H_1}\langle y_i', y_i'\rangle_{H_2} = \sum_{i=1}^n \|x_i'\|^2\|y_i'\|^2.$$

Hence, for every $i = 1, ..., n$, we have $x_i' = 0$ or $y_i' = 0$ and finally $u = \sum_{i=1}^n x_i' \otimes y_i' = 0$.

The bilinearity of $\langle\cdot,\cdot\rangle_2$ is obvious and the symmetry follows directly with the symmetry of $\langle\cdot,\cdot\rangle_{H_1}$ and $\langle\cdot,\cdot\rangle_{H_2}$. □

Definition 2.4. We denote by $H_1 \otimes_2 H_2$ the tensor product of H_1 and H_2 furnished with the scalar product $\langle \cdot, \cdot \rangle_2$ as defined above and its completion as $H_1 \widehat{\otimes}_2 H_2$. Thus, $H_1 \widehat{\otimes}_2 H_2$ is with extension of the scalar product $\langle \cdot, \cdot \rangle_2$ a Hilbert space.

The induced norm on $H_1 \widehat{\otimes}_2 H_2$ is written as $\| \cdot \|_2$.

Proposition 2.9. *Let $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$ be orthonormal bases in H_1 and H_2, respectively. Then $(e_i \otimes f_j)_{i \in I, j \in J}$ is an orthonormal basis in $H_1 \widehat{\otimes}_2 H_2$.*

Proof. It is easy to see that $(e_i \otimes f_j)_{i \in I, j \in J}$ is an orthonormal system: let $(i_1, j_1) \in I \times J$ and $(i_2, j_2) \in I \times J$ be arbitrary. Then we have

$$\begin{aligned} \langle e_{i_1} \otimes f_{j_1}, e_{i_2} \otimes f_{j_2} \rangle_2 &= \langle e_{i_1}, e_{i_2} \rangle_{H_1} \langle f_{j_1}, f_{j_2} \rangle_{H_2} \\ &= \delta_{i_1 i_2} \delta_{j_1 j_2} = \begin{cases} 1, & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ 0, & \text{if } i_1 \neq i_2 \text{ or } j_1 \neq j_2. \end{cases} \end{aligned}$$

Now we show that the elementary tensors $h_1 \otimes h_2$ with $h_1 \in H_1$ and $h_2 \in H_2$ are in the closure of $\operatorname{span}\{e_i \otimes f_j \mid i \in I, j \in J\}$. For h_1 and h_2, there are $i_1, i_2, ... \in I$ and $j_1, j_2, ... \in J$ with $h_1 = \sum_{i=1}^{\infty} \alpha_i e_i$ and $h_2 = \sum_{j=1}^{\infty} \beta_j f_j$, $\alpha_i, \beta_j \in \mathbb{R}$. Then we get

$$h_1 \otimes h_2 = (\sum_{i=1}^{\infty} \alpha_i e_i) \otimes (\sum_{j=1}^{\infty} \beta_j f_j) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_i \beta_j e_i \otimes f_j.$$

The span of the elementary tensors is by definition dense in $H_1 \widehat{\otimes}_2 H_2$ and so $\operatorname{span}\{e_i \otimes f_j \mid i \in I, j \in J\}$ is dense in $H_1 \widehat{\otimes}_2 H_2$ as well. □

Proposition 2.10. *Let H_1 and H_2 be Hilbert spaces and let E and F be Banach spaces. Moreover, let $a \in \mathfrak{L}(H_1, E)$ and $b \in \mathfrak{L}(H_2, F)$ be linear and bounded operators. Then the linear operator $a \otimes b : H_1 \otimes H_2 \to E \otimes F$ is bounded from $H_1 \otimes_2 H_2$ to $E \otimes_\epsilon F$ and we have $\|a \otimes b\| \leq \|a\| \|b\|$.*

Proof. In Proposition 2.6 we have shown $\|(a \otimes b)(u)\| \leq \|a\| \|b\| \epsilon(u)$, $u \in H_1 \otimes H_2$. Thus, it is enough to prove $\epsilon(u) \leq \|u\|_2$ for every $u = \sum_{i=1}^{n} x_i \otimes y_i \in H_1 \otimes H_2$. By the Riesz Representation Theorem, we have

$$\begin{aligned} \epsilon(u) &= \sup_{\substack{\varphi \in B_{H_1^*} \\ \psi \in B_{H_2^*}}} |\sum_{i=1}^{n} \varphi(x_i) \psi(y_i)| \\ &= \sup_{\substack{\varphi \in B_{H_1^*} \\ \psi \in B_{H_2^*}}} |\sum_{i=1}^{n} \langle x_i, J_{H_1}^{-1}(\varphi) \rangle_{H_1} \langle y_i, J_{H_2}^{-1}(\psi) \rangle_{H_2}| \\ &= \sup_{\substack{\varphi \in B_{H_1^*} \\ \psi \in B_{H_2^*}}} |\sum_{i=1}^{n} \langle x_i \otimes y_i, J_{H_1}^{-1}(\varphi) \otimes J_{H_2}^{-1}(\psi) \rangle_2|, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{H_1}$ and $\langle \cdot, \cdot \rangle_{H_2}$ shall be the scalar products of H_1 and H_2 respectively. If we take the sum into the scalar product and apply Cauchy Schwartz's inequality (cf. Inequality I.1.4 in [4]), we obtain

$$\begin{aligned}\epsilon(u) &\leq \sup_{\substack{\varphi\in B_{H_1^*}\\ \psi\in B_{H_2^*}}} \|\sum_{i=1}^{n} x_i \otimes y_i\|_2 \|J_{H_1}^{-1}(\varphi)\otimes J_{H_2}^{-1}(\psi)\|_2 \\ &= \|\sum_{i=1}^{n} x_i \otimes y_i\|_2. \qquad \square\end{aligned}$$

For the unique determined extension of $a \otimes b$ to a linear and bounded operator from $H_1 \widehat{\otimes}_2 H_2$ to $E \widehat{\otimes}_\epsilon F$, we write $a \otimes b$, too.

Chapter 3

The Result of Fernique

Our aim in this chapter is to show that the norm of Gaussian random variables with values in a normed space is squared integrable.

Definition 3.1. Let E be a separable normed space. An E-valued random variable X is called Gaussian if the real valued random variables $a(X)$ are Gaussian for every $a \in E^*$.

In fact, we show that there is a $\lambda > 0$ so that

$$\mathbb{E}\, \exp(\lambda \|X\|^2) < \infty.$$

This property is also called strong integrability. Then the squared integrability of the norm is an easy consequence.

Let $\Omega \neq \emptyset$ be an arbitrary set. For every subset $\mathfrak{Z}$ of the power set of Ω, there is a smallest σ-algebra on Ω that contains all sets of $\mathfrak{Z}$. We write $\sigma(\mathfrak{Z})$ for this σ-algebra. For a normed space E we define

$$\mathfrak{B}(E) := \sigma(\{G \subset E \mid G \text{ is open}\})$$

and say $\mathfrak{B}(E)$ is the Borel σ-algebra over E.

3.1 Characteristic Functionals

In this first section we generalize the concept of characteristic functions of a $\mathbb{R}^n$-valued random variable to random variables with values in arbitrary separable normed spaces.

Definition 3.2. Let E be a metric space. Then E fulfills the Lindelöf property if for every subset G of E with $G = \bigcup_{i \in I} G_i$, where we suppose the G_i to be open, there are $i_1, i_2, \ldots \in I$ with $G = \bigcup_{j=1}^{\infty} G_{i_j}$.

Proposition 3.1. *Every separable normed space E has the Lindelöf property.*

Proof. Let $G \subset E$ be covered by $\bigcup_{i \in I} G_i$, where $G_i \subset E$ is open for every $i \in I$. Since E is separable, there is a countable set $D := \{x_1, x_2, ...\} \subset E$, which is dense in E. We consider the countable set

$$\mathfrak{D} := \{B_{\frac{m}{n}}(x_i) \mid i, m, n \in \mathbb{N}\}.$$

For every $B \in \mathfrak{D}$, we choose an $i_0 \in I$ with $B \subset G_{i_0}$ and set $G_B := G_{i_0}$. If there is no $i_0 \in I$ with $B \subset G_{i_0}$, we set $G_B := \emptyset$. Then the countable union $\bigcup_{B \in \mathfrak{D}} G_B$ covers G: for every $x \in G$ there is an $i_x \in I$ with $x \in G_{i_x}$ and since G_{i_x} is open, there is a real $r > 0$ with $x \in B_r(x) \subset G_{i_x}$. Now we fix a $x' \in D$ with $\|x - x'\| < \frac{r}{4}$ and choose natural numbers m and n with $\frac{r}{4} < \frac{m}{n} < \frac{r}{2}$. Then we obtain $x \in B_{\frac{m}{n}}(x') \subset B_r(x) \subset G_{i_x}$. The set $G_{B_{\frac{m}{n}}(x')}$ is not empty, because one possible index in the definition of $G_{B_{\frac{m}{n}}(x')}$ is $i_0 = i_x$. Thus, we have $x \in G_{B_{\frac{m}{n}}(x')}$. This leads finally to $x \in \bigcup_{B \in \mathfrak{D}} G_B$. □

A set of the form $H(a, \alpha) := \{x \in E \mid a(x) \leq \alpha\}$ for a given $a \in E^*$ and a real number α is called a half space.

Proposition 3.2. *The Borel sets $\mathfrak{B}(E)$ are generated by half spaces.*

Proof. It is obvious that $\sigma\{H(a, \alpha) \mid a \in E^*, \alpha \in \mathbb{R}\}$ is a subset of $\mathfrak{B}(E)$. Thus, it is enough to show the second direction.

Let $D := \{x_1, x_2, ...\}$ be a countable and dense subset of E. By Hahn Banach's Theorem, there is for every $x_i \in D$ an $a_i \in E^*$ with $\|a_i\| \leq 1$ and $a_i(x_i) = \|x_i\|$. We show that $B_\varepsilon(0) = \bigcap_{i=1}^{\infty} \{x \in E \mid a_i(x) \leq \varepsilon\}$. Let x be an element of $B_\varepsilon(0)$. Then for every $i \in \mathbb{N}$, we have

$$a_i(x) \leq |a_i(x)| \leq \|a_i\| \|x\| \leq \varepsilon.$$

Thus, $B_\varepsilon(0)$ is a subset of $\bigcap_{i=1}^{\infty} \{x \in E \mid a_i(x) \leq \varepsilon\}$. Now let x_0 be an element of E with $\|x_0\| > \varepsilon$, i.e. there is a real $\delta > 0$ with $\|x_0\| = \varepsilon + \delta$. Since D is dense in E, there is an $x_i \in D$ with $\|x_0 - x_i\| \leq \delta/4$ and thus

$$a_i(x_i) = \|x_i\| \geq \|x_0\| - \|x_0 - x_i\| \geq \varepsilon + \frac{3\delta}{4}.$$

With this estimate we get

$$\begin{aligned} a_i(x_0) &= a_i(x_0) - a_i(x_i) + a_i(x_i) \\ &\geq -|a_i(x_0) - a_i(x_i)| + a_i(x_i) \\ &\geq -\|a_i\| \|x_0 - x_i\| + \varepsilon + \frac{3\delta}{4} \\ &\geq -\frac{\delta}{4} + \varepsilon + \frac{3\delta}{4} > \varepsilon. \end{aligned}$$

Hence, x_0 is not an element of $\bigcap_{i=1}^{\infty} \{x \in E \mid a_i(x) \leq \varepsilon\}$.

Thus, for every fixed $x_0 \in E$, we have

$$\begin{aligned} B_\varepsilon(x_0) &= \bigcap_{i=1}^{\infty} \{x \in E \mid a_i(x - x_0) \leq \varepsilon\} \\ &= \bigcap_{i=1}^{\infty} \{x \in E \mid a_i(x) \leq \varepsilon + a_i(x_0)\} \\ &= \bigcap_{i=1}^{\infty} H(a_i, \varepsilon + a_i(x_0)) \end{aligned}$$

and so $B_\varepsilon(x_0) \in \sigma\{H(a,\alpha) \mid a \in E^*, \alpha \in \mathbb{R}\}$.

Now let $G \subset E$ be an open subset of E. Then for every $x \in G$, there is an $\varepsilon_x > 0$ with $B_{\varepsilon_x}(x) \subset G$ and thus, $G = \bigcup_{x \in G} B_{\varepsilon_x}(x)$. The Lindelöf property yields $x^{(1)}, x^{(2)}, ... \in G$ with $G = \bigcup_{j=1}^{\infty} B_{\varepsilon_{x^{(j)}}}(x^{(j)})$. Using the results above, we have $G \in \sigma\{H(a,\alpha) \mid a \in E^*, \alpha \in \mathbb{R}\}$ which leads to the desired. □

We denote by $\sigma(a \in E^*)$ the smallest σ-algebra on E for which all $a \in E^*$ are measurable.

Corollary. *We have* $\mathfrak{B}(E) = \sigma(a \in E^*)$.

Proof. The assertion is obvious, since the Borel sets $\mathfrak{B}(E)$ are generated by half spaces and $H(a,\alpha) = a^{-1}((-\infty, \alpha])$ for every $a \in E^*$ and $\alpha \in \mathbb{R}$, i.e. $H(a,\alpha) \in \sigma(a \in E^*)$. □

Definition 3.3. For a probability measure μ on $(E, \mathfrak{B}(E))$, we define the characteristic functional as a mapping $\widehat{\mu} : E^* \to \mathbb{C}$ by

$$\widehat{\mu}(a) := \int_E \exp(ia(x)) d\mu(x).$$

Theorem 3.1. *Let μ_1 and μ_2 be two probability measures on $(E, \mathfrak{B}(E))$ with $\widehat{\mu_1}(a) = \widehat{\mu_2}(a)$ for every $a \in E^*$. Then μ_1 and μ_2 are equal on $(E, \mathfrak{B}(E))$.*

Proof. We call sets $A \subset E$ of the form

$$A = \{x \in E \mid (a_1(x), ..., a_n(x)) \in B\}, \quad n \in \mathbb{N}, a_i \in E^*, B \in \mathfrak{B}(\mathbb{R}^n)$$

cylindrical sets. We define $\mathfrak{Z}$ as

$$\mathfrak{Z} := \{A \subset E \mid A \text{ cylindrical set}\}.$$

Then $\mathfrak{Z}$ is an intersection-stable generating system of $\mathfrak{B}(E)$: by Proposition 3.2 it is clear that $\mathfrak{Z}$ is a generating system, since every half space is a cylindrical set. To see the intersection-stability, let A_1 and A_2 be cylindrical sets, i.e.

$$\begin{aligned} A_1 &= \{x \in E \mid (a_1^{(1)}(x), ..., a_n^{(1)}(x)) \in B_1\}, \quad n \in \mathbb{N},\, a_i^{(1)} \in E^*,\, B_1 \in \mathfrak{B}(\mathbb{R}^n) \\ A_2 &= \{x \in E \mid (a_1^{(2)}(x), ..., a_m^{(2)}(x)) \in B_2\}, \quad m \in \mathbb{N},\, a_i^{(2)} \in E^*,\, B_2 \in \mathfrak{B}(\mathbb{R}^m). \end{aligned}$$

Then we have

$$A_1 \cap A_2 = \{x \in E \mid (a_1^{(1)}(x), ..., a_n^{(1)}(x), a_1^{(2)}(x), ..., a_m^{(2)}(x)) \in B_1 \times B_2\}$$

and thus, $A_1 \cap A_2$ is a cylindrical set as well.

Now we show that μ_1 and μ_2 are identical on $\mathfrak{Z}$. Let A be a cylindrical set with representation $A = \{x \in E \mid (a_1(x), ..., a_n(x)) \in B\}$ and let ϑ_i be the image measure of μ_i by the mapping $J : E \to \mathbb{R}^n$ with $x \mapsto (a_1(x), ..., a_n(x))$ for $i = 1, 2$. For these measures, we have by the Transformation Theorem (cf. Proposition 2.6.5 in [3]) and with $y = (y_1, ..., y_n) \in \mathbb{R}^n$

$$\begin{aligned}
\widehat{\vartheta_1}(y) &= \int_{\mathbb{R}^n} \exp(i\langle t, y\rangle) d\vartheta_1(t) \\
&= \int_E \exp(i\langle Jx, y\rangle) d\mu_1(x) \\
&= \int_E \exp(i \sum_{k=1}^n a_k(x) y_k) d\mu_1(x) \\
&= \int_E \exp(i(\sum_{k=1}^n y_k a_k)(x)) d\mu_1(x) = \widehat{\mu_1}(\sum_{k=1}^n y_k a_k).
\end{aligned}$$

With the assumption $\widehat{\mu_1}(a) = \widehat{\mu_2}(a)$ for every $a \in E^*$, we obtain

$$\widehat{\vartheta_1}(y) = \widehat{\mu_1}(\sum_{k=1}^n y_k a_k) = \widehat{\mu_2}(\sum_{k=1}^n y_k a_k) = \widehat{\vartheta_2}(y)$$

for every $y \in \mathbb{R}^n$. By the well known unique Theorem for characteristic functions of probability measures on $\mathbb{R}^n$ (cf. Corollary 2 to Theorem 23.2 in [1]), we get the identity of ϑ_1 and ϑ_2 on $\mathfrak{B}(\mathbb{R}^n)$. In particular, we have

$$\mu_1(A) = \mu_1 \circ J^{-1}(B) = \vartheta_1(B) = \vartheta_2(B) = \mu_2(A)$$

for every cylindrical set $A \subset E$. Thus, we have the identity of μ_1 and μ_2 on an intersection stable generating system of $\mathfrak{B}(E)$ and hence, the unique Theorem for measures (cf. Corollary 1.6.2 in [3]) leads to the identity of μ_1 and μ_2 on $\mathfrak{B}(E)$. □

Corollary. *Let X and Y be two E-valued random variables, where $a(X)$ and $a(Y)$ are identical distributed for every $a \in E^*$. Then X and Y are identical distributed, i.e. the probability measures $\mathbb{P}_X$ and $\mathbb{P}_Y$ on $(E, \mathfrak{B}(E))$ induced by the random variables X and Y are equal.*

Proof. For every $a \in E^*$, the Transformation Theorem yields

$$\begin{aligned}\widehat{\mathbb{P}_X}(a) &= \int_E \exp(ia(x))d\mathbb{P}_X(x)\\ &= \int_\Omega \exp(ia(X(\omega)))d\mathbb{P}(\omega)\\ &= \mathbb{E}\exp(ia(X)).\end{aligned}$$

With the assumption, we get

$$\widehat{\mathbb{P}_X}(a) = \mathbb{E}\exp(ia(X)) = \mathbb{E}\exp(ia(Y)) = \widehat{\mathbb{P}_Y}(a)$$

and thus, by the theorem, the identity of $\mathbb{P}_X$ and $\mathbb{P}_Y$ on $(E, \mathfrak{B}(E))$. □

3.2 Fernique's Theorem

Now we show the strong integrability of a Gaussian random variable X. Let E and F be two separable normed spaces. Then $E \times F$ becomes with respect to the norm $\|(x,y)\| := \|x\| + \|y\|$ for $x \in E$ and $y \in F$ a separable normed space as well.

Proposition 3.3. *We can identify the spaces $E^* \times F^*$ and $(E \times F)^*$. A pairing is given by $J : E^* \times F^* \to (E \times F)^*$ with $J(\varphi, \psi)((x,y)) = \varphi(x) + \psi(y)$ for $\varphi \in E^*$ and $\psi \in F^*$.*

Proof. The linearity and injectivity of J is obvious. Now let $\vartheta \in (E \times F)^*$ be arbitrary. Then $\varphi(x) := \vartheta((x,0))$ for $x \in E$ defines a linear and bounded functional on E:

$$\|\varphi\| = \sup_{\substack{x\in E\\ \|x\|\le 1}} |\vartheta((x,0))| \le \sup_{\substack{x\in E\\ \|x\|\le 1}} \|\vartheta\|\|(x,0)\| = \sup_{\substack{x\in E\\ \|x\|\le 1}} \|\vartheta\|\|x\| = \|\vartheta\|.$$

In an analogous manner, $\psi(y) := \vartheta((0,y))$ for $y \in F$ defines a linear and bounded functional on F. Now we have

$$\begin{aligned}J(\varphi,\psi)((x,y)) &= \varphi(x) + \psi(y) = \vartheta((x,0)) + \vartheta((0,y))\\ &= \vartheta((x,0)+(0,y)) = \vartheta((x,y))\end{aligned}$$

for every $x \in E$ and $y \in F$. □

Theorem 3.2. *Let X be an E-valued Gaussian random variable and let X_1 and X_2 be independent copies of X. Then for every $\theta \in \mathbb{R}$ the distributions of $(\cos(\theta)X_1 - \sin(\theta)X_2, \sin(\theta)X_1 + \cos(\theta)X_2)$ and (X_1, X_2) are equal.*

Proof. Let $\theta \in \mathbb{R}$ be fixed. We show for every $\varphi \in E^*$ and $\psi \in F^*$ that the real valued random variables $J(\varphi,\psi)((\cos(\theta)X_1 - \sin(\theta)X_2, \sin(\theta)X_1 + \cos(\theta)X_2))$ and $J(\varphi,\psi)((X_1,X_2))$ are equally distributed. It is obvious that both variables are

Gaussian. So we only have to show that the variances are identically. It is easy to see that

$$\begin{aligned}
&\mathbb{E}\, J(\varphi,\psi)((\cos(\theta)X_1-\sin(\theta)X_2,\sin(\theta)X_1+\cos(\theta)X_2))^2\\
&\quad=\cos^2(\theta)\mathbb{E}\,\varphi(X_1)^2+\sin^2(\theta)\mathbb{E}\,\varphi(X_2)^2+\sin^2(\theta)\mathbb{E}\,\psi(X_1)^2+\cos^2(\theta)\mathbb{E}\,\psi(X_2)^2\\
&\quad=(\sin^2(\theta)+\cos^2(\theta))\mathbb{E}\,\varphi(X)^2+(\sin^2(\theta)+\cos^2(\theta))\mathbb{E}\,\psi(X)^2\\
&\quad=\mathbb{E}\,\varphi(X)^2+\mathbb{E}\,\psi(X)^2.
\end{aligned}$$

Note that we used the fact that X_1 and X_2 are copies of X in the third line.

On the other hand, we have

$$\begin{aligned}
\mathbb{E}\,(J(\varphi,\psi)((X_1,X_2)))^2&=\mathbb{E}\,\varphi(X_1)^2+2\mathbb{E}\,\varphi(X_1)\psi(X_2)+\mathbb{E}\,\psi(X_2)^2\\
&=\mathbb{E}\,\varphi(X)^2+\mathbb{E}\,\psi(X)^2.
\end{aligned}$$

Thus, an application of Proposition 3.3 and the Corollary to Theorem 3.1 yields the desired. □

Theorem 3.3 (Fernique)**.** *Let E be a separable normed space and let X be a Gaussian random variable with values in E. Then there is a real $\lambda > 0$ with $\mathbb{E}\,\exp(\lambda\|X\|^2) < \infty$.*

Proof. We choose a real number $s > 0$ with $p := \mathbb{P}\,(\|X\| \le s) > 1/2$ and set $t_n := (1+2^{1/2})(2^{(n+1)/2}-1)s$ for $n = 0, 1, 2...$ In particular, we have $t_0 = s$ and $t_n \nearrow \infty$ as $n \to \infty$ as well as

$$\begin{aligned}
2^{\frac12}t_n+s&=(1+2^{\frac12})(2^{\frac{n+2}{2}}-2^{\frac12})s+s\\
&=(2^{\frac{n+2}{2}}-2^{\frac12}+2^{\frac{n+3}{2}}-1)s\\
&=(1+2^{\frac12})(2^{\frac{n+2}{2}}-1)s=t_{n+1}
\end{aligned}$$

and thus, $t_n = (t_{n+1}-s)/\sqrt{2}$.

Let X_1 and X_2 be independent copies of X. Then we obtain

$$\begin{aligned}
\mathbb{P}\,(\|X\|\le s)\mathbb{P}\,(\|X\|>t_n)&=\mathbb{P}\,(\|X_1\|\le s)\mathbb{P}\,(\|X_2\|>t_n)\\
&=\mathbb{P}\,(\|X_1\|\le s,\|X_2\|>t_n)\\
&=\mathbb{P}\,(\|X_1-X_2\|\le\sqrt{2}s,\|X_1+X_2\|>\sqrt{2}t_n),
\end{aligned}$$

where we used Theorem 3.2 with $\theta = \pi/4$ in the last line.

Starting with $\|X_1 - X_2\| \le \sqrt{2}s$ and $\|X_1 + X_2\| > \sqrt{2}t_n$, we estimate

$$\begin{aligned}
t_n-s&<\frac{\|X_1+X_2\|}{\sqrt{2}}-\frac{\|X_1-X_2\|}{\sqrt{2}}\\
&\le\sqrt{2}\frac{\|X_1\|+\|X_2\|}{2}-\frac{\|X_1-X_2\|}{\sqrt{2}}.
\end{aligned}$$

We know that for all real numbers $\alpha, \beta \in \mathbb{R}$ the minimum is given by

$$\min\{\alpha, \beta\} = \frac{\alpha+\beta}{2} - \frac{|\alpha-\beta|}{2}.$$

Using this, we get

$$\begin{aligned} t_n - s &< \sqrt{2}\Big(\min\{\|X_1\|, \|X_2\|\} + \frac{|\|X_1\| - \|X_2\||}{2}\Big) - \frac{\|X_1 - X_2\|}{\sqrt{2}} \\ &\leq \sqrt{2}\min\{\|X_1\|, \|X_2\|\} + \frac{\|X_1 - X_2\|}{\sqrt{2}} - \frac{\|X_1 - X_2\|}{\sqrt{2}} \\ &= \sqrt{2}\min\{\|X_1\|, \|X_2\|\}. \end{aligned}$$

Thus, we have $t_n - s < \sqrt{2}\|X_1\|$ and $t_n - s < \sqrt{2}\|X_2\|$ which leads to

$$\begin{aligned} \mathbb{P}(\|X\| \leq s)\mathbb{P}(\|X\| > t_n) &\leq \mathbb{P}(\sqrt{2}\|X_1\| > t_n - s, \sqrt{2}\|X_2\| > t_n - s) \\ &= \mathbb{P}(\|X_1\| > \frac{t_n - s}{\sqrt{2}})\mathbb{P}(\|X_2\| > \frac{t_n - s}{\sqrt{2}}) \\ &= \mathbb{P}(\|X\| > t_{n-1})^2. \end{aligned}$$

Hence, we can keep on as follows:

$$\begin{aligned} \mathbb{P}(\|X\| > t_n) &\leq \frac{\mathbb{P}(\|X\| > t_{n-1})^2}{\mathbb{P}(\|X\| \leq s)} \leq \left(\frac{\mathbb{P}(\|X\| > t_{n-1})}{\mathbb{P}(\|X\| \leq s)}\right)^2 \\ &\leq \cdots \leq \left(\frac{\mathbb{P}(\|X\| > t_0)}{\mathbb{P}(\|X\| \leq s)}\right)^{2^n}. \end{aligned}$$

We defined $p = \mathbb{P}(\|X\| \leq s)$ and we have $t_0 = s$. So we get

$$\mathbb{P}(\|X\| > t_n) \leq (\frac{1-p}{p})^{2^n} = \exp(2^n \ln(\frac{1-p}{p})).$$

Now let $\lambda > 0$ be arbitrary. Then we have

$$\begin{aligned} \mathbb{E}\exp(\lambda\|X\|^2) &= \int_{\{\|X(\omega)\| \leq s\}} \exp(\lambda\|X(\omega)\|^2) d\mathbb{P}(\omega) \\ &\quad + \int_{\{\|X(\omega)\| > s\}} \exp(\lambda\|X(\omega)\|^2) d\mathbb{P}(\omega) \\ &\leq \exp(\lambda s^2)\mathbb{P}(\|X\| \leq s) \\ &\quad + \sum_{n=0}^{\infty} \int_{\{t_n < \|X(\omega)\| \leq t_{n+1}\}} \exp(\lambda\|X(\omega)\|^2) d\mathbb{P}(\omega). \end{aligned}$$

The first summand and the first member of the sum are only constants. So we have

$\mathbb{E}\exp(\lambda\|X\|^2) < \infty$ if

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{\{t_n < \|X(\omega)\| \leq t_{n+1}\}} \exp(\lambda \|X(\omega)\|^2) d\mathbb{P}(\omega) \\
&\qquad \leq \sum_{n=1}^{\infty} \exp(\lambda t_{n+1}^2) \mathbb{P}(\|X\| > t_n) \leq \sum_{n=1}^{\infty} \exp(\lambda t_{n+1}^2) \exp(2^n \ln(\frac{1-p}{p})) \\
&\qquad = \sum_{n=1}^{\infty} \exp(\lambda t_{n+1}^2 + 2^n \ln(\frac{1-p}{p})) < \infty.
\end{aligned}$$

This is true for every $\lambda > 0$ with

$$\lambda < \frac{\ln \frac{p}{1-p}}{4(1+\sqrt{2})^2 s^2}.$$

With such a λ we have

$$\begin{aligned}
\lambda t_{n+1}^2 + 2^n \ln(\frac{1-p}{p}) &< \frac{\ln(\frac{p}{1-p})(1+2^{\frac{1}{2}})^2 (2^{\frac{n+2}{2}} - 1)^2 s^2}{4(1+2^{\frac{1}{2}})^2 s^2} + 2^n \ln(\frac{1-p}{p}) \\
&= \ln(\frac{p}{1-p})(\frac{(2^{\frac{n+2}{2}} - 1)^2 - 2^{n+2}}{4}) \\
&= \ln(\frac{p}{1-p})(-2^{\frac{n}{2}} + \frac{1}{4}).
\end{aligned}$$

Using this estimate, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \exp(\lambda t_{n+1}^2 + 2^n \ln(\frac{1-p}{p})) \\
&\quad \leq \sum_{n=1}^{\infty} \exp(\ln(\frac{p}{1-p})(-2^{\frac{n}{2}} + \frac{1}{4})) = \sum_{n=1}^{\infty} (\frac{p}{1-p})^{-2^{\frac{n}{2}} + \frac{1}{4}} = (\frac{p}{1-p})^{\frac{1}{4}} \sum_{n=1}^{\infty} (\frac{1-p}{p})^{2^{\frac{n}{2}}} \\
&\quad \leq (\frac{p}{1-p})^{\frac{1}{4}} \sum_{n=1}^{\infty} (\frac{1-p}{p})^{\frac{n}{2}} \leq 2(\frac{p}{1-p})^{\frac{1}{4}} \sum_{n=1}^{\infty} (\frac{1-p}{p})^n = 2(\frac{p}{1-p})^{\frac{1}{4}} \frac{1}{1 - \frac{1-p}{p}} < \infty
\end{aligned}$$

and so we have

$$\mathbb{E}\exp(\lambda\|X\|^2) < \infty. \qquad \square$$

Corollary. *Let X be a Gaussian E-valued random variable. Then $\|X\|$ has a p-th moment for every $0 \leq p < \infty$.*

Proof. By Fernique's Theorem, there is a real number $\lambda > 0$ with $\mathbb{E}\exp(\lambda\|X\|^2) < \infty$. Since the exponential function grows faster than every polynomial, there is a real $c \in \mathbb{R}$ with $\|x\|^p \leq \exp(\lambda\|x\|^2)$ for every $x \in E$ with $\|x\| \geq c$. Thus, we can estimate

$$\begin{aligned}\infty &> \mathbb{E}\,\exp(\lambda\|X\|^2)\\ &= \int_{\{\|x\|\geq c\}} \exp(\lambda\|x\|^2)d\mathbb{P}_X(x) + \int_{\{\|x\|<c\}} \exp(\lambda\|x\|^2)d\mathbb{P}_X(x)\\ &\geq \int_{\{\|x\|\geq c\}} \|x\|^p d\mathbb{P}_X(x).\end{aligned}$$

The term $\int_{\{\|x\|<c\}} \|x\|^p d\mathbb{P}_X(x)$ is lower than a second constant $c' \in \mathbb{R}$ and so we get

$$\mathbb{E}\,\|X\|^p \leq c' + \int_{\{\|x\|\geq c\}} \|x\|^p d\mathbb{P}_X(x) < \infty.$$ □

Chapter 4

The Result of Chevet

This chapter is about the convergence of random series with values in separable Banach spaces. In the following, let $(\xi_i)_{i=1}^\infty$ and $(\xi_{ij})_{i,j=1}^\infty$ be independent sequences of standard normal random variables, defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Moreover, let $(x_i)_{i=1}^\infty$ and $(y_j)_{j=1}^\infty$ be series in real separable Banach spaces E and F, respectively.

4.1 Preliminaries

We define the Gaussian and the weak norm of a sequence $(x_i)_{i=1}^\infty \subset E$ and summarize a few properties of these numbers.

Definition 4.1. For $0 < p < \infty$ and an arbitrary E-valued random variable η, we define the Gaussian norm $\|\eta\|_p$ as

$$\|\eta\|_p := (\mathbb{E}\,\|\eta\|^p)^{\frac{1}{p}}$$

and for series $(x_i)_{i=1}^\infty \subset E$, we set

$$\|(x_i)_{i=1}^\infty\|_p = \|(x_i)_{i=1}^\infty\|_{p,E} := \sup_n \|\sum_{i\leq n} \xi_i x_i\|_p \ .$$

Finally, we define the weak norm $M_2[(x_i)_{i=1}^\infty]$ as

$$M_2[(x_i)_{i=1}^\infty] := \sup_{\varphi\in B_{E^*}} \left(\sum_{i=1}^\infty |\varphi(x_i)|^2\right)^{\frac{1}{2}} .$$

Note that we allow the value infinite for the Gaussian norm as well as for the weak norm.

Proposition 4.1. *Let n be a natural number and $\alpha_1, ..., \alpha_n \in \mathbb{R}$. Then $\sum_{i\leq n} \xi_i\alpha_i$ and $\xi_1(\sum_{i\leq n} \alpha_i^2)^{1/2}$ are identically distributed. In particular, this implies for $0 <$*

$p < \infty$

$$\| \sum_{i \le n} \xi_i \alpha_i \|_p = \|\xi_1\|_p \left(\sum_{i \le n} \alpha_i^2 \right)^{\frac{1}{2}} .$$

Proof. We set $\eta_i := \xi_i \alpha_i$ and obtain $\sum_{i \le n} \eta_i \sim \mathcal{N}(0, \sum_{i \le n} \alpha_i^2)$. Thus, we have

$$\frac{\sum_{i \le n} \eta_i}{\sqrt{\sum_{i \le n} \alpha_i^2}} = \frac{\sum_{i \le n} \xi_i \alpha_i}{\sqrt{\sum_{i \le n} \alpha_i^2}} \sim \mathcal{N}(0, 1),$$

which yields

$$\sum_{i \le n} \xi_i \alpha_i \sim \xi_1 \left(\sum_{i \le n} \alpha_i^2 \right)^{\frac{1}{2}} .$$

Using the first part leads to

$$\| \sum_{i \le n} \xi_i \alpha_i \|_p = \left(\mathbb{E} \, |\xi_1 (\sum_{i \le n} \alpha_i^2)^{1/2}|^p \right)^{\frac{1}{p}} = \|\xi_1\|_p \left(\sum_{i \le n} \alpha_i^2 \right)^{\frac{1}{2}} . \qquad \square$$

Proposition 4.2. *Let η be an arbitrary E-valued random variable, $1 \le p < \infty$, $x \in E$ and ξ a real valued symmetric random variable, which is assumed to be independent from η. Then we have*

$$\left(\mathbb{E} \, \|\eta\|^p \right)^{\frac{1}{p}} \le \left(\mathbb{E} \, \|\eta + \xi x\|^p \right)^{\frac{1}{p}} .$$

Proof. By Fubini's Theorem (cf. Theorem 5.2.2 in [3]) and the symmetry of ξ we get

$$\begin{aligned} \left(\mathbb{E} \, \|\eta + \xi x\|^p \right)^{\frac{1}{p}} &= \left(\int_{E \times \mathbb{R}} \|t_1 + t_2 x\|^p d\mathbb{P}_{\eta,\xi}(t_1, t_2) \right)^{\frac{1}{p}} \\ &= \left(\int_E \int_0^\infty \|t_1 + t_2 x\|^p + \|t_1 - t_2 x\|^p d\mathbb{P}_{\xi}(t_2) d\mathbb{P}_{\eta}(t_1) \right)^{\frac{1}{p}} . \end{aligned}$$

Now note that, by the convexity of $\| \cdot \|^p$ for $p \ge 1$, we have

$$\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \ge 2\|x_1\|^p.$$

for all $x_1, x_2 \in E$. Hence, we obtain

$$\begin{aligned} \left(\mathbb{E} \, \|\eta + \xi x\|^p \right)^{\frac{1}{p}} &\ge \left(\int_E \int_0^\infty 2\|t_1\|^p d\mathbb{P}_{\xi}(t_2) d\mathbb{P}_{\eta}(t_1) \right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty 2 d\mathbb{P}_{\xi}(t_2) \int_E \|t_1\|^p d\mathbb{P}_{\eta}(t_1) \right)^{\frac{1}{p}} \\ &= \left(\mathbb{E} \, \|\eta\|^p \right)^{\frac{1}{p}} . \end{aligned} \qquad \square$$

Note that this Proposition yields that the sequence $(\mathbb{E}\,\|\sum_{i\le n}\xi_i x_i\|^p)_{n=1}^\infty$ is monotonic increasing for $1\le p<\infty$ and $(x_i)_{i=1}^\infty\subset E$.

Proposition 4.3. *For every $0<p<\infty$, we have*

$$M_2[(x_i)_{i=1}^\infty]\le \beta_p\|(x_i)_{i=1}^\infty\|_p \quad \textit{with} \quad \beta_p=(\mathbb{E}\,|\xi_1|^p)^{-\frac{1}{p}}=\|\xi_1\|_p^{-1}.$$

Proof. Using Proposition 4.1, we get

$$\begin{aligned} M_2[(x_i)_{i=1}^\infty)] &= \sup_{\varphi\in B_{E^*}}\sup_n\left(\sum_{i\le n}|\varphi(x_i)|^2\right)^{\frac{1}{2}} \\ &= \sup_{\varphi\in B_{E^*}}\sup_n\frac{\|\sum_{i\le n}\xi_i\varphi(x_i)\|_p}{\|\xi_1\|_p} \\ &= \beta_p\sup_{\varphi\in B_{E^*}}\sup_n\left(\mathbb{E}\,|\sum_{i\le n}\xi_i\varphi(x_i)|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Now we use the linearity of φ and the fact $|\varphi(x)|\le\|\varphi\|\|x\|$ and obtain the assertion

$$\begin{aligned} M_2[(x_i)_{i=1}^\infty)] &\le \beta_p\sup_{\varphi\in B_{E^*}}\sup_n\left(\mathbb{E}\,\|\varphi\|^p\|\sum_{i\le n}\xi_i x_i\|^p\right)^{\frac{1}{p}} \\ &= \beta_p\sup_n\left(\mathbb{E}\,\|\sum_{i\le n}\xi_i x_i\|^p\right)^{\frac{1}{p}} \\ &= \beta_p\|(x_i)_i\|_p. \end{aligned}$$

□

Note that $\beta_2=1$. Because of this, we often omit it.

Proposition 4.4. *If $\sum_{i=1}^\infty\xi_i x_i$ converges a.s. in E, then we have for every $1\le p<\infty$*

$$\|(x_i)_{i=1}^\infty\|_p=\left(\mathbb{E}\left\|\sum_{i=1}^\infty\xi_i x_i\right\|^p\right)^{\frac{1}{p}}.$$

Proof. This Proposition is an easy consequence of Proposition 4.2. □

Proposition 4.5. *Let $(x_i)_{i=1}^\infty\subset E$ be a sequence with $M_2[(x_i)_{i=1}^\infty]<\infty$ and let $G\subseteq B_{E^*}$ be a norming subset. Then, for every $0<p<\infty$,*

$$\|(x_i)_{i=1}^\infty\|_p=\left(\mathbb{E}\sup_{\varphi\in G}\left|\sum_{i=1}^\infty\xi_i\varphi(x_i)\right|^p\right)^{\frac{1}{p}}.$$

Proof. Firstly, since G is assumed to be a norming set, we have

$$\|(x_i)_{i=1}^\infty\|_p=\sup_n\left(\mathbb{E}\sup_{\varphi\in G}|\sum_{i\le n}\xi_i\varphi(x_i)|^p\right)^{\frac{1}{p}}.$$

We know by Proposition 4.2 that $\mathbb{E}\sup_{\varphi\in G}|\sum_{i\leq n}\xi_i\varphi(x_i)|^p$ is monotonic increasing with increasing n. Thus, it is enough to show that the right hand side of the equation is less than infinite. This can be seen as follows: by Proposition 4.1, we have

$$\begin{aligned}\left(\mathbb{E}\sup_{\varphi\in G}|\sum_{i\leq n}\xi_i\varphi(x_i)|^p\right)^{\frac{1}{p}} &= \sup_{\varphi\in G}(\sum_{i\leq n}|\varphi(x_i)|^2)^{\frac{1}{2}}\,(\mathbb{E}\,|\xi_1|^p)^{\frac{1}{p}}\\ &= M_2[(x_i)_{i=1}^n]\,(\mathbb{E}\,|\xi_1|^p)^{\frac{1}{p}}\\ &\leq M_2[(x_i)_{i=1}^\infty]\,(\mathbb{E}\,|\xi_1|^p)^{\frac{1}{p}}\,.\end{aligned}$$

Since this is true for every natural number n and we assumed $M_2[(x_i)_{i=1}^\infty]<\infty$, we obtain as $n\to\infty$

$$\|(x_i)_{i=1}^\infty\|_p=\left(\mathbb{E}\sup_{\varphi\in G}\left|\sum_{i=1}^\infty\xi_i\varphi(x_i)\right|^p\right)^{\frac{1}{p}}. \qquad \square$$

A very important and well-known result in the field of random series is the following

Theorem 4.1 (Lévy)**.** *Let $\eta_1,\eta_2,\ldots$ be independent symmetric E-valued random variables. Then the following statements are equivalent:*

i. $(\sum_{i\leq n}\eta_i)_{n=1}^\infty$ *converges a.s.*

ii. $(\sum_{i\leq n}\eta_i)_{n=1}^\infty$ *converges in probability.*

Proof. We refer to Theorem 2.1.1 in [9]. $\square$

Finally, we state the Theorem of Sudakov and Fernique, where we also only give a reference for the proof.

Theorem 4.2 (Sudakov-Fernique)**.** *Let I be an arbitrary nonempty set and let $(\eta(s))_{s\in I}$ and $(\xi(s))_{s\in I}$ be families of $\mathbb{R}^n$-valued Gaussian random vectors. Then*

$$\mathbb{E}\,|\eta(s)-\eta(t)|^2\leq\mathbb{E}\,|\xi(s)-\xi(t)|^2$$

for all $s,t\in I$ implies for $0<p<\infty$

$$\mathbb{E}\sup_{(s,t)\in I\times I}|\eta(s)-\eta(t)|^p\leq\mathbb{E}\sup_{(s,t)\in I\times I}|\xi(s)-\xi(t)|^p.$$

Proof. We refer to Theorem 2.1.2 in [5]. $\square$

4.2 Chevet's Theorem

The result of Chevet is of vital importance for this thesis. For reasons of clarity, we split it into two Theorems.

Theorem 4.3. *For* $(x_i)_{i=1}^\infty \subset E$, $(y_j)_{j=1}^\infty \subset F$ *and for every real* $1 \le p < \infty$, *we have*

$$\|(x_i \otimes y_j)_{i,j=1}^\infty\|_{p,E\widehat{\otimes}_\epsilon F} \le \sqrt{2}A_p$$

with

$$A_p = M_2[(x_i)_{i=1}^\infty]\|(y_j)_{j=1}^\infty\|_p + M_2[(y_j)_{j=1}^\infty]\|(x_i)_{i=1}^\infty\|_p.$$

Proof. Without loss of generality we assume $\|(x_i)_{i=1}^\infty\|_p < \infty$ and $\|(y_j)_{j=1}^\infty\|_p < \infty$. Otherwise the inequality is trivial. Firstly, we check the inequality only for finite sequences $(x_i)_{i=1}^n$ and $(y_j)_{j=1}^n$ and use a limiting argument at the end of the proof. Let us consider random variables $P_n : \Omega \to E\widehat{\otimes}_\epsilon F$ and $Q_n : \Omega \times \Omega \to E \times F$ with

$$\begin{aligned} P_n(\omega) &= \sum_{i,j=1}^n \xi_{ij}(\omega) x_i \otimes y_j \quad \text{and} \\ Q_n(\omega,\omega') &= (Q_{n,1}(\omega), Q_{n,2}(\omega')) \\ &= \sqrt{2}\sum_{i=1}^n (M_2[(y_k)_{k=1}^n]\xi_i(\omega)x_i; M_2[(x_k)_{k=1}^n]\xi_i(\omega')y_i). \end{aligned}$$

Note that $\{\varphi\otimes\psi \mid \varphi \in B_{E^*},\ \psi \in B_{F^*}\} \subset B_{(E\widehat{\otimes}_\epsilon F)^*}$ is a norming subset for $E\widehat{\otimes}_\epsilon F$, by definition of the ϵ-norm. Thus, by Proposition 4.5, we get

$$\begin{aligned} \|(x_i \otimes y_j)_{i,j=1}^n\|_{p,E\widehat{\otimes}_\epsilon F} &= \left(\mathbb{E} \sup_{\substack{\|\varphi\|\le 1\\ \|\psi\|\le 1}} |\sum_{i,j=1}^n \xi_{ij}(\varphi\otimes\psi)(x_i\otimes y_j)|^p\right)^{\frac{1}{p}} \\ &= \left(\mathbb{E} \sup_{(\varphi,\psi)\in I} |(\varphi\otimes\psi)(P_n)|^p\right)^{\frac{1}{p}}. \end{aligned}$$

with $I := B_{E^*\times F^*}$.

In order to apply Theorem 4.2 for $((\varphi\otimes\psi)(P_n))_{(\varphi,\psi)\in I}$ and $((\varphi,\psi)(Q_n))_{(\varphi,\psi)\in I}$, we verify the requirements of this Theorem. For all (φ_1,ψ_1) and $(\varphi_2,\psi_2) \in I$ we have

$$\begin{aligned} &\mathbb{E}\,|(\varphi_1\otimes\psi_1)(P_n) - (\varphi_2\otimes\psi_2)(P_n)|^2 \\ &= \mathbb{E}\,|\sum_{i,j=1}^n \xi_{ij}((\varphi_1\otimes\psi_1)(x_i\otimes y_j) - (\varphi_2\otimes\psi_2)(x_i\otimes y_j))|^2 \\ &= \sum_{i,j=1}^n |\varphi_1(x_i)\psi_1(y_j) - \varphi_2(x_i)\psi_2(y_j)|^2, \end{aligned}$$

where we used Proposition 4.1 and $(\varphi\otimes\psi)(x\otimes y) = \varphi(x)\psi(y)$. The insertion of

$\pm\varphi_2(x_i)\psi_1(y_j)$ in the first and the convexity of $|\cdot|^2$ in the second line leads to

$$\begin{aligned}
&\mathbb{E}\,|(\varphi_1 \otimes \psi_1)(P_n) - (\varphi_2 \otimes \psi_2)(P_n)|^2 \\
&\quad = \sum_{i,j=1}^{n} |(\varphi_1 - \varphi_2)(x_i)\psi_1(y_j) + (\psi_1 - \psi_2)(y_j)\varphi_2(x_i)|^2 \\
&\quad \leq 2\sum_{i,j=1}^{n} |(\varphi_1 - \varphi_2)(x_i)\psi_1(y_j)|^2 + 2\sum_{i,j=1}^{n} |(\psi_1 - \psi_2)(y_j)\varphi_2(x_i)|^2.
\end{aligned}$$

The separation of the sums and a reapplication of Proposition 4.1 yields

$$\begin{aligned}
&\mathbb{E}\,|(\varphi_1 \otimes \psi_1)(P_n) - (\varphi_2 \otimes \psi_2)(P_n)|^2 \\
&\quad \leq 2\sum_{i=1}^{n} |(\varphi_1 - \varphi_2)(x_i)|^2 \sum_{j=1}^{n} |\psi_1(y_j)|^2 \\
&\qquad\qquad + 2\sum_{j=1}^{n} |(\psi_1 - \psi_2)(y_j)|^2 \sum_{i=1}^{n} |\varphi_2(x_i)|^2 \\
&\quad \leq 2\mathbb{E}\,|\sum_{i=1}^{n} \xi_i(\omega)(\varphi_1 - \varphi_2)(x_i)|^2 M_2^2[(y_j)_{j=1}^n] \\
&\qquad\qquad + 2\mathbb{E}\,|\sum_{j=1}^{n} \xi_j(\omega')(\psi_1 - \psi_2)(y_j)|^2 M_2^2[(x_i)_{i=1}^n].
\end{aligned}$$

Now, we also take the expectation over $\sqrt{2}M_2[(x_i)_{i=1}^n]$ and $\sqrt{2}M_2[(y_j)_{j=1}^n]$ and obtain by the linearity of $(\varphi_1 - \varphi_2)$ and $(\psi_1 - \psi_2)$

$$\begin{aligned}
&\mathbb{E}\,|(\varphi_1 \otimes \psi_1)(P_n) - (\varphi_2 \otimes \psi_2)(P_n)|^2 \\
&\quad \leq \mathbb{E}\,|(\varphi_1 - \varphi_2)(\sqrt{2}\sum_{i=1}^{n} M_2[(y_j)_{j=1}^n]\xi_i(\omega)x_i)|^2 \\
&\qquad\qquad + \mathbb{E}\,|(\psi_1 - \psi_2)(\sqrt{2}\sum_{j=1}^{n} M_2[(x_i)_{i=1}^n]\xi_j(\omega')y_j)|^2 \\
&\quad = \mathbb{E}\,|(\varphi_1 - \varphi_2)(Q_{n,1})|^2 + \mathbb{E}\,|(\psi_1 - \psi_2)(Q_{n,2})|^2 \\
&\quad = \mathbb{E}\,|(\varphi_1, \psi_1)(Q_n) - (\varphi_2, \psi_2)(Q_n)|^2.
\end{aligned}$$

Hence, we can apply Theorem 4.2 and get

$$\begin{aligned}
&\mathbb{E} \sup_{\substack{(\varphi_1,\psi_1)\in I \\ (\varphi_2,\psi_2)\in I}} |(\varphi_1 \otimes \psi_1)(P_n) - (\varphi_2 \otimes \psi_2)(P_n)|^p \\
&\qquad\qquad \leq \mathbb{E} \sup_{\substack{(\varphi_1,\psi_1)\in I \\ (\varphi_2,\psi_2)\in I}} |(\varphi_1, \psi_1)(Q_n) - (\varphi_2, \psi_2)(Q_n)|^p.
\end{aligned}$$

Note that I is a symmetric subset of $B_{E^* \times F^*}$. Therefore, the equation above is equal to

$$\mathbb{E} \sup_{(\varphi,\psi)\in I} |2(\varphi \otimes \psi)(P_n)|^p \leq \mathbb{E} \sup_{(\varphi,\psi)\in I} |2(\varphi,\psi)(Q_n)|^p$$

and thus,

$$\begin{aligned} \left(\mathbb{E} \sup_{(\varphi,\psi)\in I} |(\varphi \otimes \psi)(P_n)|^p\right)^{\frac{1}{p}} &\leq \left(\mathbb{E} \sup_{(\varphi,\psi)\in I} |(\varphi,\psi)(Q_n)|^p\right)^{\frac{1}{p}} \\ &= \left(\mathbb{E} \sup_{(\varphi,\psi)\in I} |(\varphi(Q_{n,1})^2 + \psi(Q_{n,2})^2)^{\frac{1}{2}}|^p\right)^{\frac{1}{p}}. \end{aligned}$$

The simple estimate $(\varphi(Q_{n,1})^2 + \psi(Q_{n,2})^2)^{\frac{1}{2}} \leq |\varphi(Q_{n,1})| + |\psi(Q_{n,2})|$ yields

$$\begin{aligned} \left(\mathbb{E} \sup_{(\varphi,\psi)\in I} |(\varphi \otimes \psi)(P_n)|^p\right)^{\frac{1}{p}} &\leq \left(\mathbb{E} \Big| \sup_{\varphi\in B_{E^*}} |\varphi(Q_{n,1})| + \sup_{\psi\in B_{F^*}} |\psi(Q_{n,2})|\Big|^p\right)^{\frac{1}{p}} \\ &= \left(\mathbb{E} \big| \|Q_{n,1}\| + \|Q_{n,2}\|\big|^p\right)^{\frac{1}{p}}, \end{aligned}$$

where we used Hahn Banach's Theorem in the last equation. We apply the Minkowski inequality (cf. Proposition 3.3.3 in [3]) and by the definition of Q_n we get

$$\begin{aligned} \left(\mathbb{E} \sup_{(\varphi,\psi)\in I} |(\varphi \otimes \psi)(P_n)|^p\right)^{\frac{1}{p}} &\leq (\mathbb{E} \|Q_{n,1}\|^p)^{\frac{1}{p}} + (\mathbb{E} \|Q_{n,2}\|^p)^{\frac{1}{p}} \\ &= \sqrt{2}\big(M_2[(y_k)_{k=1}^n] \|(x_i)_{i=1}^n\|_p + M_2[(x_k)_{k=1}^n] \|(y_j)_{j=1}^n\|_p\big) \\ &\leq \sqrt{2} A_p, \end{aligned}$$

which leads to

$$\begin{aligned} \|(x_i \otimes y_j)_{i,j=1}^n\|_{p, E \widehat{\otimes}_\epsilon F} &= \left(\mathbb{E} \sup_{(\varphi,\psi)\in I} |(\varphi \otimes \psi)(P_n)|^p\right)^{\frac{1}{p}} \\ &\leq \sqrt{2} A_p. \end{aligned}$$

This estimate holds true for every $n \in \mathbb{N}$ and thus, we obtain the assertion

$$\|(x_i \otimes y_j)_{i,j=1}^\infty\|_{p, E \widehat{\otimes}_\epsilon F} \leq \sqrt{2} A_p .$$

□

Theorem 4.4 (Chevet)**.** *Let $(x_i)_{i=1}^\infty \subset E$ and $(y_j)_{j=1}^\infty \subset F$ be sequences in E and F. If $\sum_{i=1}^\infty \xi_i x_i$ and $\sum_{j=1}^\infty \xi_j y_j$ converge a.s. in E and F, respectively, then $\sum_{i,j=1}^\infty \xi_{ij} x_i \otimes y_j$ converges a.s. in $E \widehat{\otimes}_\epsilon F$.*

Proof. Let $\sum_{i=1}^\infty \xi_i x_i$ and $\sum_{j=1}^\infty \xi_j y_j$ be a.s. convergent in E and F respectively. Then, by Fernique's Theorem (cf. the Corollary to Theorem 3.3), there is a real

$c < \infty$ with $\|(x_i)_{i=1}^\infty\|_2 < c$ and $\|(y_j)_{j=1}^\infty\|_2 < c$. Moreover, there is for every real $\varepsilon > 0$ a natural number n_0 with

$$\|(x_i)_{i=n_0}^\infty\|_2 = \left(\mathbb{E}\,\|\sum_{i=n_0}^{\infty}\xi_i x_i\|^2\right)^{\frac{1}{2}} < \varepsilon \qquad \text{and}$$

$$\|(y_j)_{j=n_0}^\infty\|_2 = \left(\mathbb{E}\,\|\sum_{j=n_0}^{\infty}\xi_j y_j\|^2\right)^{\frac{1}{2}} < \varepsilon.$$

With these results we can estimate

$$\left(\mathbb{E}\,\|\sum_{i,j=1}^{m}\xi_{ij}x_i\otimes y_j - \sum_{i,j=1}^{n-1}\xi_{ij}x_i\otimes y_j\|^2\right)^{\frac{1}{2}} < 2\sqrt{2}\varepsilon(\varepsilon+2c)$$

for all $m \geq n-1 \geq n_0$. In order to prove this assertion we define for simplification $I_1(m,n)$, $I_2(m,n)$ and $I_3(m,n)$ as

$$I_1(m,n) = \sum_{i=n}^{m}\sum_{j=n}^{m}\xi_{ij}x_i\otimes y_j, \quad I_2(m,n) = \sum_{i=1}^{n-1}\sum_{j=n}^{m}\xi_{ij}x_i\otimes y_j \quad \text{and}$$

$$I_3(m,n) = \sum_{i=n}^{m}\sum_{j=1}^{n-1}\xi_{ij}x_i\otimes y_j.$$

Using these definitions, we obtain by Minkowski's inequality in the second line

$$\begin{aligned}&\left(\mathbb{E}\,\|\sum_{i,j=1}^{m}\xi_{ij}x_i\otimes y_j - \sum_{i,j=1}^{n-1}\xi_{ij}x_i\otimes y_j\|^2\right)^{\frac{1}{2}}\\ &\quad = \left(\mathbb{E}\,\|I_1(m,n)+I_2(m,n)+I_3(m,n)\|^2\right)^{\frac{1}{2}}\\ &\quad \leq \left(\mathbb{E}\,\|I_1(m,n)\|^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\,\|I_2(m,n)\|^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\,\|I_3(m,n)\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

For the first summand we have by Proposition 4.2

$$\begin{aligned}\left(\mathbb{E}\,\|I_1(m,n)\|^2\right)^{\frac{1}{2}} &\leq \sup_{m'\geq n_0}\left(\mathbb{E}\,\|I_1(m',n_0)\|^2\right)^{\frac{1}{2}}\\ &= \|(x_i\otimes y_j)_{i,j=n_0}^\infty\|_2.\end{aligned}$$

Now we apply Theorem 4.3 and get, by Proposition 4.3 in the second line,

$$\begin{aligned}\left(\mathbb{E}\,\|I_1(m,n)\|^2\right)^{\frac{1}{2}} &\leq \sqrt{2}\left(M_2[(x_i)_{i=n_0}^\infty]\|(y_j)_{j=n_0}^\infty\|_2 + M_2[(y_j)_{j=n_0}^\infty]\|(x_i)_{i=n_0}^\infty\|_2\right)\\ &\leq 2\sqrt{2}\|(x_i)_{i=n_0}^\infty\|_2\|(y_j)_{j=n_0}^\infty\|_2 < 2\sqrt{2}\varepsilon^2.\end{aligned}$$

For the second summand we use similar arguments. Firstly, we have by Proposition

4.2

$$\left(\mathbb{E}\,\|I_2(m,n)\|^2\right)^{\frac{1}{2}} \le \sup_{m'\ge n_0}\left(\mathbb{E}\,\|\sum_{i=1}^{m'}\sum_{j=n_0}^{m'}\xi_{ij}x_i\otimes y_j\|^2\right)^{\frac{1}{2}}$$
$$= \|(x_i\otimes y_j)_{i=1,j=n_0}^{\infty}\|_2.$$

Again, by Theorem 4.3 and Proposition 4.3, we get

$$\left(\mathbb{E}\,\|I_2(m,n)\|^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\|(x_i)_{i=1}^{\infty}\|_2\|(y_j)_{j=n_0}^{\infty}\|_2 < 2\sqrt{2}c\varepsilon.$$

In an analogous manner we obtain for the third summand

$$\left(\mathbb{E}\,\|I_3(m,n)\|^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\|(x_i)_{i=n_0}^{\infty}\|_2\|(y_j)_{j=1}^{\infty}\|_2 < 2\sqrt{2}\varepsilon c$$

and thus,

$$\left(\mathbb{E}\,\|\sum_{i,j=1}^{m}\xi_{ij}x_i\otimes y_j - \sum_{i,j=1}^{n-1}\xi_{ij}x_i\otimes y_j\|^2\right)^{\frac{1}{2}} < 2\sqrt{2}\varepsilon^2 + 2\sqrt{2}c\varepsilon + 2\sqrt{2}\varepsilon c.$$

Hence, $\sum_{i,j=1}^{\infty}\xi_{ij}x_i\otimes y_j$ converges in L_2. By Tschebyscheff's inequality (cf. Equality 3.18 in [1]) we obtain for every real $\lambda > 0$

$$\mathbb{P}\,(\|\sum_{i,j=n}^{\infty}\xi_{ij}x_i\otimes y_j\| > \lambda) \le \frac{\mathbb{E}\,\|\sum_{i,j=n}^{\infty}\xi_{ij}x_i\otimes y_j\|^2}{\lambda^2}$$
$$\to 0$$

as $n\to\infty$. Thus, $\sum_{i,j=1}^{\infty}\xi_{ij}x_i\otimes y_j$ converges in probability and finally, by Theorem 4.1, almost surely. □

Chapter 5

Operator Generated Processes

For Gaussian processes with almost surely continuous paths, we find in this chapter series representations like the ones that appeared in the last chapter. Thus, Chevet's Result is applicable to our problem: is the tensor product of Gaussian processes with continuous paths a process with continuous paths as well?

In the following let $u^* : E^* \to F^*$ always be the dual operator of a linear and bounded operator $u : E \to F$, where E and F are arbitrary normed spaces.

5.1 Definition and Properties

Proposition 5.1. *Let (S, ρ) be a compact metric space and let u be a linear and bounded operator from a Hilbert space H into $C(S)$. Then the series*

$$\sum_{i=1}^{\infty} \xi_i u(e_i)(s)$$

converges a.s. for every orthonormal basis $(e_i)_{i=1}^{\infty} \subset H$ and every $s \in S$. But note that the set of convergence depends on $s \in S$.

Proof. Let J_H be the isometrical isomorphism that identifies the Hilbert space H with its dual. By Parseval's identity (cf. Theorem I.4.13 in [4]), we have

$$\infty > \|J_H^{-1} u^* \delta_s\|_H^2 = \sum_{i=1}^{\infty} |\langle J_H^{-1} u^* \delta_s, e_i \rangle_H|^2 = \sum_{i=1}^{\infty} |u^* \delta_s(e_i)|^2 = \sum_{i=1}^{\infty} |u(e_i)(s)|^2.$$

Thus, there is for every $\varepsilon > 0$ a natural number n_0 with

$$\sum_{i=n+1}^{m} |u(e_i)(s)|^2 < \varepsilon$$

for all $m > n \geq n_0$. This estimation yields

$$\begin{aligned}\mathbb{E}\,|\sum_{i=1}^{m}\xi_i u(e_i)(s) - \sum_{i=1}^{n}\xi_i u(e_i)(s)|^2 &= \sum_{i=n+1}^{m}\mathbb{E}\,|\xi_i u(e_i)(s)|^2 \\ &= \sum_{i=n+1}^{m}|u(e_i)(s)|^2 < \varepsilon.\end{aligned}$$

Hence, $(\sum_{i=1}^{n}\xi_i u(e_i)(s))_{n=1}^{\infty}$ is a Cauchy sequence in L_2. Because of the completeness of L_2 the series $\sum_{i=1}^{\infty}\xi_i u(e_i)(s)$ converges in L_2; by Tschebyscheff's inequality in probability and finally, by Theorem 4.1, almost surely. □

Definition 5.1. Obviously, by Proposition 5.1, an operator $u : H \to C(S)$ induces a Gaussian process $X = (X_s)_{s\in S}$ with

$$X_s := \sum_{i=1}^{\infty}\xi_i u(e_i)(s).$$

We say that this process is generated by the operator u.

Proposition 5.2. *Let a Gaussian process $X = (X_s)_{s\in S}$ be generated by an operator $u : H \to C(S)$. Then the covariance function $R_X : S \times S \to \mathbb{R}$ of X is represented by*

$$R(s_1, s_2) := \mathbb{E}\,X_{s_1}X_{s_2} = \langle J_H^{-1}u^*\delta_{s_1}, J_H^{-1}u^*\delta_{s_2}\rangle_H$$

for all $s_1, s_2 \in S$. In particular, it follows that a change of the orthonormal basis in Definition 5.1 leads to a version of X.

Proof. We start with the definition of a generated process, i.e. we have

$$X_s = \sum_{i=1}^{\infty}\xi_i u(e_i)(s)$$

for an orthonormal basis $(e_i)_{i=1}^{\infty}$. With this representation we obtain

$$\begin{aligned}\mathbb{E}\,X_{s_1}X_{s_2} &= \mathbb{E}\sum_{i=1}^{\infty}\xi_i u(e_i)(s_1)\sum_{j=1}^{\infty}\xi_j u(e_j)(s_2) \\ &= \mathbb{E}\sum_{i=1}^{\infty}\xi_i\langle J_H^{-1}u^*\delta_{s_1}, e_i\rangle_H\sum_{j=1}^{\infty}\xi_j\langle J_H^{-1}u^*\delta_{s_2}, e_j\rangle_H \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{E}\,\xi_i\xi_j\langle J_H^{-1}u^*\delta_{s_1}, e_i\rangle_H\langle J_H^{-1}u^*\delta_{s_2}, e_j\rangle_H.\end{aligned}$$

We know that $\mathbb{E}\,\xi_i\xi_j = 1$ if i is equal to j and 0 otherwise. Thus, we are able to replace $\mathbb{E}\,\xi_i\xi_j$ by $\langle e_i, e_j\rangle_H$ and get

$$\begin{aligned}\mathbb{E}\,X_{s_1}X_{s_2} &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle e_i, e_j\rangle_H \langle J_H^{-1}u^*\delta_{s_1}, e_i\rangle_H \langle J_H^{-1}u^*\delta_{s_2}, e_j\rangle_H \\ &= \langle \sum_{i=1}^{\infty}\langle J_H^{-1}u^*\delta_{s_1}, e_i\rangle_H e_i, \langle \sum_{j=1}^{\infty}\langle J_H^{-1}u^*\delta_{s_2}, e_j\rangle_H e_j\rangle_H \\ &= \langle J_H^{-1}u^*\delta_{s_1}, J_H^{-1}u^*\delta_{s_2}\rangle_H,\end{aligned}$$

(cf. Theorem I.4.13 in [4] for the last line) which is the desired equation. □

5.2 Bochner's Integral

In the following, we need the Bochner integral as a generalization of Lebesgue's integral to vector valued mappings. We only give a short introduction and refer to Appendix E in [3] for details and proofs.

For an arbitrary set Ω and a subset $A \subset \Omega$ we define a mapping $\mathbb{I}_A : \Omega \to \{0,1\}$ as

$$\mathbb{I}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

for every $\omega \in \Omega$.

Definition 5.2. Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space and let E be a separable Banach space. A random variable $X : \Omega \to E$ is called simple if there is a representation

$$X = \sum_{i=1}^{n} x_i \mathbb{I}_{A_i}$$

with $n \in \mathbb{N}$, $x_i \in E$ and $A_i \in \mathfrak{A}$.

For simple random variables we define Bochner's integral by

$$B\text{–}\int_\Omega X(\omega) d\mathbb{P}(\omega) := \sum_{i=1}^{n} x_i \mathbb{P}(A_i).$$

The following are easy consequences of the definition and of the triangle inequality of the norm.

Proposition 5.3. *For simple random variables $X, Y : \Omega \to E$ the Bochner integral fulfills*

$$B\text{–}\int_\Omega (X(\omega) + Y(\omega)) d\mathbb{P}(\omega) = B\text{–}\int_\Omega X(\omega) d\mathbb{P}(\omega) + B\text{–}\int_\Omega Y(\omega) d\mathbb{P}(\omega)$$

and

$$\|B\text{–}\int_\Omega X(\omega) d\mathbb{P}(\omega)\| \le \int_\Omega \|X(\omega)\| d\mathbb{P}(\omega).$$

Definition 5.3. We call an arbitrary random variable $X : \Omega \to E$, where E is a separable Banach space, Bochner integrable if the mapping $\omega \mapsto \|X(\omega)\|$ is (Lebesgue-) integrable.

The following Proposition enables us to define Bochner's integral for arbitrary mappings (cf. Proposition E.2 in [3]).

Proposition 5.4. *Let X be a Bochner integrable random variable. Then there is a sequence $(X_n)_{n=1}^{\infty}$ of simple random variables so that we have with*

$$X(\omega) = \lim_{n\to\infty} X_n(\omega)$$

and

$$\|X_n(\omega)\| \le \|X(\omega)\|$$

for every $n \in \mathbb{N}$ and $\omega \in \Omega$.

For a Bochner integrable random variable X, we choose a sequence $(X_n)_{n=1}^{\infty}$ like the one above. By Lebesgue's Theorem about dominated convergence (cf. Theorem 2.4.4 in [3]), we have

$$\lim_{n\to\infty} \int_{\Omega} \|X_n(\omega) - X(\omega)\| d\mathbb{P}(\omega) = 0$$

and thus,

$$\begin{aligned}
&\lim_{m,n\to\infty} \int_{\Omega} \|X_m(\omega) - X_n(\omega)\| d\mathbb{P}(\omega) \\
&\quad \le \lim_{m\to\infty} \int_{\Omega} \|X_m(\omega) - X(\omega)\| d\mathbb{P}(\omega) + \lim_{n\to\infty} \int_{\Omega} \|X(\omega) - X_n(\omega)\| d\mathbb{P}(\omega) \\
&\quad = 0.
\end{aligned}$$

Hence,

$$\lim_{m,n\to\infty} \|B\text{–}\int_{\Omega} X_m(\omega) - X_n(\omega) d\mathbb{P}(\omega)\| = 0$$

and so $(B\text{–}\int_{\Omega} X_n(\omega) d\mathbb{P}(\omega))_{n=1}^{\infty}$ is a Cauchy sequence in E. Since E is complete this, sequence converges. Now we define

$$B\text{–}\int_{\Omega} X(\omega) d\mathbb{P}(\omega) := \lim_{n\to\infty} B\text{–}\int_{\Omega} X_n(\omega) d\mathbb{P}(\omega).$$

It is easy to see that this definition is independent from the selection of the sequence $(X_n)_{n=1}^{\infty}$. We need one very important result about the Bochner integral (cf. Proposition E.11 in [3]).

Theorem 5.1. *Let E be a separable Banach space and let $X : \Omega \to E$ be a Bochner integrable random variable. Then we have for every $a \in E^*$*

$$\int_{\Omega} a(X(\omega)) d\mathbb{P}(\omega) = a(B\text{–}\int_{\Omega} X(\omega) d\mathbb{P}(\omega)).$$

5.3 The weak-* Topology

In the following, we want the unit ball B_{E^*} of the dual space of a separable normed space E to be compact. But it is well-known that this is generally not true in the topology that is induced by the norm on E^*. Thus, we introduce a weaker topology on E^* - the so-called weak-* topology.

Definition 5.4. We summarize the most important terms in the field of topological spaces.

i. A subset $\mathfrak{O}$ of the power set over a non-empty set E is called a topology on E if we have $\emptyset$ and $E \in \mathfrak{O}$, for two sets $O_1, O_2 \in \mathfrak{O}$ the intersection $O_1 \cap O_2$ is in $\mathfrak{O}$ and for an arbitrary family $\{O_i\}_{i\in I} \subset \mathfrak{O}$ the union $\bigcup_{i\in I} O_i$ is an element of $\mathfrak{O}$. The pair $(E, \mathfrak{O})$ is called a topological space.

ii. An element $O \in \mathfrak{O}$ is called an open set.

iii. A sequence $(x_n)_{n=1}^{\infty} \subset E$ converges to $x \in E$ if for every open set O with $x \in O$ there is a natural number n_0 with $x_n \in O$ for every $n \geq n_0$.

iv. A mapping $f : E_1 \to E_2$ from a topological space $(E_1, \mathfrak{O}_1)$ into another topological space $(E_2, \mathfrak{O}_2)$ is said to be continuous if $f^{-1}(O)$ is open in $\mathfrak{O}_1$ for every $O \in \mathfrak{O}_2$.

v. A mapping $f : E_1 \to E_2$ is said to be sequence-continuous if for every $x \in E_1$ and all sequences $(x_n)_{n=1}^{\infty} \subset E_1$ that converges to x the sequence $(f(x_n))_{n=1}^{\infty} \subset E_2$ converges to $f(x) \in E_2$.

vi. A subset G of E is said to be compact if for every cover $G \subset \bigcup_{i\in I} O_i$ with $O_i \in \mathfrak{O}$ there is a finite selection of indices $i_1, i_2, ..., i_n \in I$ with $G \subset \bigcup_{j=1}^{n} O_{i_j}$. The topological space $(E, \mathfrak{O})$ is called compact if E is compact.

Now let us turn to the definition of the weak-* topology.

Definition 5.5. Let E be a normed space. A subset O of E^* is said to be open in the weak-* topology if there are for every $a_0 \in O$ elements $x_1, ..., x_n \in E$ and a real number $\varepsilon > 0$ with

$$\bigcap_{j=1}^{n} \{a \in E^* : |a(x_j) - a_0(x_j)| < \varepsilon\} \subset O.$$

Proposition 5.5. *The weak-* topology is a topology on* E^*.

Proof. It is obvious that E^* and $\emptyset$ are open in the weak-* topology. Now let O_1 and O_2 be open. Then, for every $a_0 \in O_1 \cap O_2$ there are elements $x_1^{(1)}, ..., x_n^{(1)} \in E$ and

$x_1^{(2)}, ..., x_m^{(2)} \in E$ as well as real numbers $\varepsilon_1, \varepsilon_2 > 0$ with

$$\bigcap_{j=1}^{n} \left\{ a \in E^* : |a(x_j^{(1)}) - a_0(x_j^{(1)})| < \varepsilon_1 \right\} \cap \bigcap_{j=1}^{m} \left\{ a \in E^* : |a(x_j^{(2)}) - a_0(x_j^{(2)})| < \varepsilon_2 \right\} \subset O_1 \cap O_2$$

and thus $O_1 \cap O_2$ is open in the weak-* topology. Finally, let O_i, $i \in I$, be open. For every $a_0 \in \bigcup_{i \in I} O_i$ there is an i_0 with $a_0 \in O_{i_0}$ and so there are elements $x_1, ..., x_n \in E$ and a real $\varepsilon > 0$ with

$$\bigcap_{j=1}^{n} \{ a \in E^* : |a(x_j) - a_0(x_j)| < \varepsilon \} \subset O_{i_0} \subset \bigcup_{i \in I} O_i.$$

Hence, $\bigcup_{i \in I} O_i$ is also open in the weak-* topology, which completes the proof. □

Theorem 5.2 (Alaoglu)**.** *Let E be a separable Banach space. Then the set*

$$B_{E^*} := \{ a \in E^* : \|a\| \le 1 \}$$

is compact and metrizable in the weak- topology.*

Proof. We refer to Theorem V.3.1 and Theorem V.5.1 in [4]. □

With this result B_{E^*} becomes a metric space. This is very important, since a mapping $f : B_{E^*} \to M$ is now continuous if and only if it is sequence-continuous, where M is another metric space.

Proposition 5.6. *A sequence $(a_n)_{n=1}^{\infty} \subset E^*$ converges to $a \in E^*$ in the weak-* topology (written as $a_n \xrightarrow{w} a$) if and only if for every $x \in E$ the sequence $(a_n(x))_{n=1}^{\infty} \subset \mathbb{R}$ converges to $a(x) \in \mathbb{R}$.*

Proof. Let $(a_n)_{n=1}^{\infty} \subset E^*$ be a sequence in E^* and $a_0 \in E^*$ with $a_n \xrightarrow{w} a_0$. For $x \in E$ and $\varepsilon > 0$ we consider the open set

$$O = \{ a \in E^* : |a(x) - a_0(x)| < \varepsilon \}.$$

Obviously, we have $a_0 \in O$ and, since $a_n \xrightarrow{w} a_0$ as $n \to \infty$, there is a natural number n_0 with $a_n \in O$ for all $n \ge n_0$. Hence, for every $x \in E$ and $\varepsilon > 0$, there is a $n_0 \in \mathbb{N}$ with $|a_n(x) - a_0(x)| < \varepsilon$, i.e. $a_n(x) \to a_0(x)$ as $n \to \infty$ for every $x \in E$.

Now we assume $a_n(x) \to a_0(x)$ as $n \to \infty$ for every $x \in E$. Let O be open with $a_0 \in O$. We have to show that there is a natural number n_0 with $a_n \in O$ for every $n \ge n_0$. Since $a_0 \in O$, there are by definition of the weak-* topology elements $x_1, ..., x_m \in E$ and a real number $\varepsilon > 0$ with

$$O_1 := \bigcap_{j=1}^{m} \{ a \in E^* : |a(x_j) - a_0(x_j)| < \varepsilon \} \subset O.$$

With the assumption there is a natural number $n_0 \in \mathbb{N}$ so that we have $|a_n(x_j) - a_0(x_j)| < \varepsilon$ for every $n \geq n_0$ and $j = 1, ..., m$. Thus, we get $a_n \in O_1 \subset O$ for all $n \geq n_0$. □

5.4 The Reproducing Hilbert Space

Our aim in this section is to show that Gaussian processes with a.s. continuous paths are generated by an operator. In fact, we prove a more general result and get the desired as a special case.

Proposition 5.7. *Let E be a separable Banach space. Then there is a countable separating subset in E^*.*

Proof. Let $D := \{x_1, x_2, ...\}$ be dense in E. For $x_i \neq x_j$, there is by Hahn Banach's Theorem a functional $a_{ij} \in E^*$ with $\|a_{ij}\| = 1$ and $\|a_{ij}(x_i - x_j)\| = \|x_i - x_j\| > 0$. Thus, the countable set $G := \{a_{ij} \mid i, j \in \mathbb{N}\} \subset E^*$ is separating for D. We show that G is also separating for E. Let $x, y \in E$ be arbitrary elements with $x \neq y$, i.e. $\|x - y\| = \delta > 0$. We choose $x_i, x_j \in D$ so that $\|x - x_i\| < \delta/4$ and $\|y - x_j\| < \delta/4$. Then we have

$$\begin{aligned}\delta/2 &< \|x - y\| - \|x - x_i\| - \|y - x_j\| \\ &\leq \|x_i - x_j\| \\ &= \|a_{ij}(x_i - x_j)\| \\ &\leq \|a_{ij}(x_i - x)\| + \|a_{ij}(x - y)\| + \|a_{ij}(y - x_j)\| \\ &\leq \|x_i - x\| + \|a_{ij}(x - y)\| + \|y - x_j\| \\ &\leq \delta/4 + \|a_{ij}(x - y)\| + \delta/4.\end{aligned}$$

Hence, we get $\|a_{ij}(x - y)\| > 0$ which leads to $a_{ij}(x) \neq a_{ij}(y)$. □

Theorem 5.3. *Let E be a separable Banach space and let X be a Gaussian random variable with values almost surely in E. Then there is a separable Hilbert space H and an operator $u : H \to E$ with*

$$X = \sum_{i=1}^{\infty} \xi_i u(e_i)$$

almost surely for an orthonormal basis $(e_i)_{i=1}^{\infty} \subset H$.

Proof. Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be the probability space on which X is defined. We set

$$H_X^0 := \{a(X) \in L_2 \mid a \in E^*\} \subset L_2(\Omega)$$

and $H_X := \overline{H_X^0} \subset L_2$. Since H_X is a closed subspace of a Hilbert space, it is a Hilbert space as well.

In the first step we prove that H_X is separable. We consider the set

$$B := \{a \in E^* : \|a\| \leq 1\}$$

furnished with the weak-* topology. We show that the mapping $a \mapsto a(X)$ is continuous from B to L_2. Since B is a metric space, we can work with the sequence-continuity. Let $(a_n)_{n=1}^{\infty}$ converge in the weak-* topology to a, i.e. by Proposition 5.6, we have $a_n(X(\omega)) \to a(X(\omega))$ as $n \to \infty$ for every $\omega \in \Omega$. We want to apply Lebesgue's Theorem about dominated convergence to show

$$\mathbb{E}\,|(a_n - a)(X)|^2 \to 0$$

as $n \to \infty$. For this purpose, we need an integrable majorant. For every $\omega \in \Omega$ we have with the binomial equation

$$\begin{aligned} |(a_n - a)(X(\omega))|^2 &\leq |a_n(X(\omega))|^2 + 2|a_n(X(\omega))||a(X(\omega))| + |a(X(\omega))|^2 \\ &\leq \|a_n\|^2\|X(\omega)\|^2 + 2\|a_n\|\|X(\omega)\|\|a\|\|X(\omega)\| + \|a\|^2\|X(\omega)\|^2. \end{aligned}$$

Since all a_n and a are elements of B, we have $\|a_n\|, \|a\| \leq 1$ and thus,

$$|(a_n - a)X(\omega)|^2 \leq 4\|X(\omega)\|^2.$$

By Fernique's Theorem we know $\mathbb{E}\,\|X\|^2 < \infty$. Hence, $4\|X\|^2$ is an integrable majorant and Lebesgue's Theorem yields

$$\mathbb{E}\,|(a_n - a)(X)|^2 \to 0$$

as $n \to \infty$.

By Alaoglu's Theorem (cf. Theorem 5.2) we know that B is compact in the weak-* topology. Therefore, $\{a(X) : \|a\| \leq 1\}$ is as a continuous image of a compact set compact itself and thus, it is separable. Since this is not only true for $\|a\| \leq 1$, but also for $\|a\| \leq n$ for every natural number n, we have the separability of

$$H_X^0 = \bigcup_{n=1}^{\infty} \{a(X) : \|a\| \leq n\}.$$

Thus, H_X is as the closure of H_X^0 separable as well.

In the second step we show the existence of a sequence $(h_i)_{i=1}^{\infty} \subset H_X^0$, which is an orthonormal basis of H_X. Since H_X is a separable Hilbert space, there is a countable orthonormal basis $(h_i')_{i=1}^{\infty}$ in H_X (cf. Proposition I.4.16 [4]). Because of the density of H_X^0 in H_X, every h_i' has a representation $h_i' = \sum_{j=1}^{\infty} h_{i_j}$, where the h_{i_j} are elements in H_X^0. We consider the countable set $D := \{h_{i_j} \mid i, j \in \mathbb{N}\} \subset H_X^0$. Now we apply the Gram-Schmidt process on a linear independent subset of D, which spans the same subspace as D itself and obtain the desired sequence $(h_i)_{i=1}^{\infty} \subset H_X^0$.

For every h_i there is an $a_i \in H_X^0$ so that $h_i = a_i(X)$. Thus, h_i is normal distributed since X is Gaussian. We also have $\mathbb{E}\,h_i h_j = 1$ if $i = j$ and zero otherwise.

Hence, $(h_i)_{i=1}^\infty$ is an independent sequence of standard normal distributed random variables.

With Bochner's integral we define an operator $u : H_X \to E$ by

$$u(h) := B\text{–}\int_\Omega h(\omega)X(\omega)d\mathbb{P}(\omega).$$

This integral is only well defined if $\|hX\|$ is integrable. To see this, let $h \in H_X$ be arbitrary. Then there is a sequence $(h_n)_{n=1}^\infty \subset H_X^0$ with $h_n \to h$ as $n \to \infty$ in L_2. So we have by Fatou's Lemma (cf. Theorem 2.4.3 in [3])

$$\begin{aligned}\int_\Omega \|h(\omega)X(\omega)\|d\mathbb{P}(\omega) &= \int_\Omega \lim_{n\to\infty} |h_n(\omega)|\|X(\omega)\|d\mathbb{P}(\omega)\\ &\le \liminf_{n\to\infty} \int_\Omega |h_n(\omega)|\|X(\omega)\|d\mathbb{P}(\omega).\end{aligned}$$

For every $h_n \in H_X^0$ there is an $a_n \in E^*$ with $h_n(\omega) = a_n(X(\omega))$, $\omega \in \Omega$. Hence, we obtain

$$\begin{aligned}\int_\Omega \|h(\omega)X(\omega)\|d\mathbb{P}(\omega) &\le \liminf_{n\to\infty} \int_\Omega |a_n(X(\omega))|\|X(\omega)\|d\mathbb{P}(\omega)\\ &\le \liminf_{n\to\infty} \int_\Omega \|a_n\|\|X(\omega\|\|X(\omega)\|d\mathbb{P}(\omega)\\ &\le \liminf_{n\to\infty} \int_\Omega \|X(\omega)\|^2 d\mathbb{P}(\omega).\end{aligned}$$

Finally, by Fernique's Theorem, we get

$$\int_\Omega \|h(\omega)X(\omega)\|d\mathbb{P}(\omega) \le \mathbb{E}\,\|X\|^2 < \infty.$$

Now we show $X = \sum_{i=1}^\infty h_i u(h_i)$ a.s., which would complete the proof. For every $a \in E^*$ we have

$$\begin{aligned}a(X) &= \sum_{i=1}^\infty h_i \langle h_i, a(X)\rangle\\ &= \sum_{i=1}^\infty h_i \int_\Omega h_i(\omega)a(X(\omega))d\mathbb{P}(\omega).\end{aligned}$$

Using Theorem 5.1 we get

$$\begin{aligned}a(X) &= \sum_{i=1}^\infty h_i a(B\text{–}\int_\Omega h_i(\omega)X(\omega)d\mathbb{P}(\omega))\\ &= \sum_{i=1}^\infty h_i a(u(h_i)) = a(\sum_{i=1}^\infty h_i u(h_i)).\end{aligned}$$

Note that the equalities take place in $L_2(\Omega)$, i.e. for almost all $\omega \in \Omega$ we have $a(X(\omega)) = a(\sum_{i=1}^{\infty} h_i(\omega)u(h_i))$, where (in general) the exclusion set N_a depends on $a \in E^*$. By Proposition 5.7 there is countable and separating subset $D \subset E^*$. We set $N := \bigcup_{a\in D} N_a$ and since this is a countable union, we get $\mathbb{P}(N) = 0$. Thus, for almost all $\omega \in \Omega$ we have $a(X(\omega)) = a(\sum_{i=1}^{\infty} h_i(\omega)u(h_i))$ for all functionals a in a separating subset of E^*. This finally leads to $X = \sum_{i=1}^{\infty} h_i u(h_i)$ almost surely. □

Theorem 5.4. *Let (S,ρ) be a compact metric space and $X = (X_s)_{s\in S}$ a Gaussian process. Moreover, let $X : s \mapsto X(s)(\omega)$ be continuous for almost all $\omega \in \Omega$. Then there is a separable Hilbert space H and an operator $u : H \to C(S)$ with*

$$X_s = \sum_{i=1}^{\infty} \xi_i u(e_i)(s)$$

for an orthonormal basis $(e_i)_{i=1}^{\infty} \subset H$ and all $s \in S$ as well as the a.s. convergence of the series

$$\sum_{i=1}^{\infty} \xi_i u(e_i)$$

in $C(S)$.

Proof. We can understand the Gaussian process $(X_s)_{s\in S}$ as a Gaussian random variable X with values almost surely in the separable Banach space $C(S)$, where we set $X(s) = X_s$. In order to see this, we have to show for every $\lambda \in C^*(S) = \mathcal{M}(S)$ that the real valued random variable $\lambda(X)$ is Gaussian. We construct decompositions $(Z_n)_{n=1}^{\infty}$ of the compact set S in the following way: for $n \in \mathbb{N}$ we consider the covering $\bigcup_{s\in S} B_{\frac{1}{n}}(s)$ of S. Then there are elements $s_1, ..., s_{m_n}$ with $S = \bigcup_{i=1}^{m_n} B_{\frac{1}{n}}(s_i)$. Now we define the sets $S_1^{(n)}, ..., S_{m_n}^{(n)}$ as $S_i^{(n)} = B_{\frac{1}{n}}(s_i) \setminus \bigcup_{j=1}^{i-1} B_{\frac{1}{n}}(s_j)$. The resulting sets are disjunct and cover S. So $Z_n := \{S_1^{(n)}, ..., S_{m_n}^{(n)}\}$ is a decomposition of S. For two fixed mappings $\Gamma_H^{(n)} : Z_n \to S$ with $\Gamma_H^{(n)}(S_i^{(n)}) \in S_i^{(n)}$ and $\Gamma_R^{(n)} : S \to Z_n$ with $\Gamma_R^{(n)}(s) = S_{i_0}^{(n)}$ if and only if $s \in S_{i_0}^{(n)}$, we have with the construction of the decompositions $\Gamma_H^{(n)}(\Gamma_R^{(n)}(s)) \to s$ as $n \to \infty$. We define for every $s \in S$ and $n \in \mathbb{N}$ a random variable $X_s^{(n)} = X_{\Gamma_H^{(n)}(\Gamma_R^{(n)}(s))}$ and set $X^{(n)} = (X_s^{(n)})_{s\in S}$.

Now for every $n \in \mathbb{N}$ and every $\lambda \in \mathcal{M}(S)$ we obtain

$$\lambda(X^{(n)}) = \int_S X_s^{(n)} d\lambda(s) = \sum_{i=1}^{m_n} \lambda(S_i^{(n)}) X_{\Gamma_H^{(n)}(S_i^{(n)})}$$

and thus, $\lambda(X^{(n)})$ is Gaussian, because every linear combination of Gaussian random variables is Gaussian as well.

By Lebesgue's Theorem we get for every $t \in \mathbb{R}$

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E} \exp(it\lambda(X^{(n)})) &= \lim_{n\to\infty} \int_\Omega \exp(it \int_S X_s^{(n)}(\omega) d\lambda(s)) d\mathbb{P}(\omega) \\ &= \int_\Omega \exp(it \lim_{n\to\infty} \int_S X_s^{(n)}(\omega) d\lambda(s)) d\mathbb{P}(\omega).\end{aligned}$$

Since for almost every fixed $\omega \in \Omega$ the mapping $s \mapsto X_s(\omega)$ is continuous and S is compact, there is a real $c > 0$ with $\sup_{s\in S} |X_s(\omega)| < c$. Thus, a second application of Lebesgue's theorem yields

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E} \exp(it\lambda(X^{(n)})) &= \int_\Omega \exp(it \int_S \lim_{n\to\infty} X_s^{(n)}(\omega) d\lambda(s)) d\mathbb{P}(\omega) \\ &= \int_\Omega \exp(it \int_S X_s(\omega) d\lambda(s)) d\mathbb{P}(\omega) \\ &= \mathbb{E} \exp(it\lambda(X)).\end{aligned}$$

Hence, the sequence $(\lambda(X^{(n)}))_{n=1}^\infty$ converges in distribution to $\lambda(X)$.

On the other hand, the limit η of real valued Gaussian random variables η_n, $n \in \mathbb{N}$ is always Gaussian: let $(\exp(-t^2\sigma_n^2/2))_n$ be the sequence of the values of the characteristic functions of η_n, $n \in \mathbb{N}$, in point $t \in \mathbb{R}$. Since $(\eta_n)_{n=1}^\infty$ converges in distribution, this sequence is a Cauchy and thus, the sequence $(\sigma_n^2)_{n=1}^\infty$ is Cauchy, too. Hence, there is a real $\sigma^2 > 0$ with $\sigma_n^2 \to \sigma^2$ as $n \to \infty$ and we get $\exp(-t^2\sigma_n^2/2) \to \exp(-t^2\sigma^2/2)$, which is the characteristic function of a Gaussian distribution as well. Putting the last two results together we obtain that $\lambda(X)$ is Gaussian distributed for every $\lambda \in \mathcal{M}(S)$.

For the Gaussian random variable X we have by Theorem 5.3 a separable Hilbert space H and an operator $u : H \to C(S)$ with

$$X = \sum_{i=1}^\infty \xi_i u(e_i)$$

for an orthonormal basis $(e_i)_{i=1}^\infty \subset H$ and

$$X_s = X(s) = \sum_{i=1}^\infty \xi_i u(e_i)(s),$$

i.e. the process $(X_s)_{s\in S}$ is generated by the operator $u : H \to C(S)$. □

Chapter 6

Continuity of the Tensor Product Process

Proposition 6.1. *Let (S,ρ_1) and (T,ρ_2) be two compact metric spaces, H_1 and H_2 Hilbert spaces, as well as $u_1 : H_1 \to C(S)$ and $u_2 : H_2 \to C(T)$ linear and bounded operators. Then we have*

$$J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^* \delta_{(s,t)} = (J^{-1}_{H_1} u_1^* \delta_s) \otimes (J^{-1}_{H_2} u_2^* \delta_t)$$

for every $s \in S$ and $t \in T$, where $J_{H_1 \widehat{\otimes}_2 H_2}$, J_{H_1} and J_{H_2} shall be the isometric isomorphisms that identify the corresponding Hilbert space with its dual (cf. Theorem 2.2).

Proof. In the first step we show $\delta_{(s,t)} = \delta_s \otimes \delta_t$ for every $s \in S$ and $t \in T$. For every $f \in C(S)$ and $g \in C(T)$ we can understand $f \otimes g$ as a continuous real-valued mapping on $S \times T$ with $(s,t) \mapsto f(s)g(t)$ (cf. Theorem 2.1) and so we have

$$(\delta_s \otimes \delta_t)(f \otimes g) = \delta_s(f)\delta_t(g) = f(s)g(t) = (f \otimes g)(s,t) = \delta_{(s,t)}(f \otimes g).$$

By the linearity of $\delta_s \otimes \delta_t$ and $\delta_{(s,t)}$ we get $\delta_{(s,t)} = \delta_s \otimes \delta_t$.

With this result we obtain for every $h_1 \in H_1$ and $h_2 \in H_2$

$$\begin{aligned}\langle J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^* \delta_{(s,t)}, h_1 \otimes h_2 \rangle_2 &= \langle J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^*(\delta_s \otimes \delta_t), h_1 \otimes h_2 \rangle_2 \\ &= (\delta_s \otimes \delta_t)(u_1 \otimes u_2)(h_1 \otimes h_2).\end{aligned}$$

We use the definition of the tensor product operator twice and obtain

$$\begin{aligned}\langle J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^* \delta_{(s,t)}, h_1 \otimes h_2 \rangle_2 &= \delta_s(u_1 h_1)\delta_t(u_2 h_2) \\ &= (u_1^* \delta_s)(h_1)(u_2^* \delta_t)(h_2).\end{aligned}$$

Applying the definition of J_{H_1} and J_{H_2} as well as the definition of the scalar product of $H_1 \widehat{\otimes}_2 H_2$ leads to

$$\begin{aligned}\langle J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^* \delta_{(s,t)}, h_1 \otimes h_2 \rangle_2 &= \langle J_1^{-1} u_1^* \delta_s, h_1 \rangle_{H_1} \langle J_2^{-1} u_2^* \delta_t, h_2 \rangle_{H_2} \\ &= \langle (J^{-1}_{H_1} u_1^* \delta_s) \otimes (J^{-1}_{H_2} u_2^* \delta_t), h_1 \otimes h_2 \rangle_2.\end{aligned}$$

By the linearity and continuity of the scalar product we get for every $u \in H_1 \widehat{\otimes}_2 H_2$

$$\langle J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^* \delta_{(s,t)}, u \rangle_2 = \langle (J^{-1}_{H_1} u_1^* \delta_s) \otimes (J^{-1}_{H_2} u_2^* \delta_t), u \rangle_2$$

and thus,

$$J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^* \delta_{(s,t)} = (J^{-1}_{H_1} u_1^* \delta_s) \otimes (J^{-1}_{H_2} u_2^* \delta_t).$$ □

Theorem 6.1. *Let (S, ρ_1) and (T, ρ_2) be two compact metric spaces, as well as $X = (X_s)_{s \in S}$ and $Y = (Y_t)_{t \in T}$ two Gaussian processes with index sets S and T, respectively, that are not identical zero. Then $X \otimes Y$ has a.s. continuous paths if and only if X and Y has a.s. continuous paths as well.*

Proof. Firstly, let us assume that X and Y has a.s. continuous paths. With this assumption, Theorem 5.4 yields separable Hilbert spaces H_1 and H_2 as well as operators $u_1 : H_1 \to C(S)$ and $u_2 : H_2 \to C(T)$ so that

$$\sum_{i=1}^{\infty} \xi_i u_1(e_i) \qquad \text{and} \qquad \sum_{j=1}^{\infty} \xi_j u_2(f_j)$$

converges a.s. in $C(S)$ and $C(T)$, respectively, where $(e_i)_{i=1}^{\infty} \subset H_1$ and $(f_j)_{j=1}^{\infty} \subset H_2$ are orthonormal bases. Moreover, we have

$$X_s = \sum_{i=1}^{\infty} \xi_i u_1(e_i)(s) \qquad \text{as well as} \qquad X_t = \sum_{j=1}^{\infty} \xi_j u_2(f_j)(t)$$

for every $s \in S$ and $t \in T$

Now we apply Chevet's Theorem (cf. Theorem 4.4) and obtain the a.s. convergence of

$$\sum_{i,j=1}^{\infty} \xi_{ij} u_1(e_i) \otimes u_2(f_j) = \sum_{i,j=1}^{\infty} \xi_{ij} (u_1 \otimes u_2)(e_i \otimes f_j)$$

in $C(S) \widehat{\otimes}_\epsilon C(T)$ which is, by Theorem 2.1, isometric isomorph to $C(S \times T)$. Thus, the Gaussian process $Z = (Z_{(s,t)})_{(s,t) \in S \times T}$, which is generated by $u_1 \otimes u_2 : H_1 \widehat{\otimes}_2 H_2 \to C(S \times T)$ has a.s. continuous paths. We show that Z is a version of $X \otimes Y$.

The covariance function R_Z of Z is with Proposition 5.2 given by

$$R_Z((s_1, t_1), (s_2, t_2)) = \langle J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^* \delta_{(s_1,t_1)}, J^{-1}_{H_1 \widehat{\otimes}_2 H_2}(u_1 \otimes u_2)^* \delta_{(s_2,t_2)} \rangle_2.$$

By Proposition 6.1, we have

$$\begin{aligned} R_Z((s_1,t_1),(s_2,t_2)) &= \langle (J_{H_1}^{-1}u_1^*\delta_{s_1})\otimes(J_{H_2}^{-1}u_2^*\delta_{t_1}),(J_{H_1}^{-1}u_1^*\delta_{s_2})\otimes(J_{H_2}^{-1}u_2^*\delta_{t_2})\rangle_2 \\ &= \langle J_{H_1}^{-1}u_1^*\delta_{s_1},J_{H_1}^{-1}u_1^*\delta_{s_2}\rangle_{H_1}\langle J_{H_2}^{-1}u_2^*\delta_{t_1},J_{H_2}^{-1}u_2^*\delta_{t_2}\rangle_{H_2}. \end{aligned}$$

Using Proposition 5.2 a second time we obtain

$$R_Z((s_1,t_1),(s_2,t_2)) = R_X(s_1,s_2)R_Y(t_1,t_2)$$

and thus, we get the desired, i.e. the Gaussian process with almost surely continuous paths Z is a version of $X\otimes Y$.

On the other hand, let us assume that $Z := X\otimes Y$ has a.s. continuous paths. Fix a $t_0\in T$ with $\mathbb{E}\,Y_{t_0}^2>0$. This is possible since we assumed the process Y not to be identical zero. Then the paths of $(Z_{(s,t_0)})_{s\in S}$ are a.s. continuous and thus, the paths of $((\mathbb{E}\,Y_{t_0}^2)^{-1/2}Z_{(s,t_0)})_{s\in S}$ as well. For the covariance function $R_{t_0}: S\times S\to\mathbb{R}$ of this Gaussian process we have by the definition of the tensor product process for all $s_1,s_2\in S$

$$\begin{aligned} R_{t_0}(s_1,s_2) &= \mathbb{E}\left[(\mathbb{E}\,Y_{t_0}^2)^{-1/2}Z_{(s_1,t_0)}(\mathbb{E}\,Y_{t_0}^2)^{-1/2}Z_{(s_2,t_0)}\right] \\ &= (\mathbb{E}\,Y_{t_0}^2)^{-1}\mathbb{E}\,X_{s_1}X_{s_2}\mathbb{E}\,Y_{t_0}Y_{t_0} \\ &= \mathbb{E}\,X_{s_1}X_{s_2}. \end{aligned}$$

Hence, $((\mathbb{E}\,Y_{t_0}^2)^{-1/2}Z_{(s,t_0)})_{s\in S}$ is a version of X with a.s. continuous paths. The existence of a version of Y with a.s. continuous paths is shown in an analogous manner. Therefore, the proof is complete. □

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Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Jena, 16.06.2008